Models of Markov processes with a random transition mechanism

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Abstract. The paper deals with a certain class of random evolutions. We develop a construction that yields an invariant measure for a continuous-time Markov process with random transitions. The approach is based on a particular way of constructing the combined process, where the generator is defined as a sum of two terms: one responsible for the evolution of the environment and the second representing generators of processes with a given state of environment. (The two operators are not assumed to commute.) The presentation includes fragments of a general theory and pays a particular attention to several types of examples: (1) a queueing system with a random change of parameters (including a Jackson network and, as a special case: a single-server queue with a diffusive behavior of arrival and service rates), (2) a simple-exclusion model in presence of a special 'heavy' particle, (3) a diffusion with drift-switching, and (4) a diffusion with a randomly diffusion-type varying diffusion coefficient (including a modification of the Heston random volatility model).

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1 Introduction

This paper presents a construction of Markovian models of random processes in a random environment. The problem can be stated in the following form. Suppose we are given a family of 'basic'

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continuous-time Markov processes (MPs) $\widetilde{X}^{(z)}(t)$ on a state space $\mathbb{X}$ where z is a parameter describing a 'state of environment' (SE) which can be varied within a space $\mathcal{Z}$. Assume that $\forall\, z \in \mathcal{Z}$, process $\widetilde{X}^{(z)}$ has an invariant measure (IM) $\nu^{(z)}$. Further, suppose that we have an MP $\widetilde{Z}(t)$ on $\mathcal{Z}$ with an IM υ. Is it possible to construct a 'combined' (two-component) MP $(Z(t), X(t))$ on $\mathcal{Z}\times\mathbb{X}$ in a 'universal' manner, allowing a natural interpretation of an MP in a random environment? In this paper we put forward such a construction (under certain limitations); a feature of our constructions is a mutual impact of an SE and a state of a basic process upon each other. In other words, we describe a general mechanism where the state of a basic process influences a change of an environment which in turn leads to a new transition rule for the process. Such an approach should be compared with a body of work on random walks in random environment (cf., e.g., [20]) where an environment is randomly chosen but kept fixed throughout the time dynamics.

An example of a system to which our construction can be applied is a Jackson network (JN) model; see [9], [10]. Here the environment can be identified with a triple $(\underline{\lambda}, \underline{\mu}, \mathtt{P})$ where $\underline{\lambda}$ and $\underline{\mu}$ are vectors of arrival and service rates, and $\mathtt{P}$ is a routing matrix. The IM is a product of geometric distributions identified in terms of $\underline{\lambda}$, $\underline{\mu}$ and $\mathtt{P}$. Making triple $(\underline{\lambda}, \underline{\mu}, \mathtt{P})$ dependent on parameter z varying randomly within a finite or countable set $\mathcal{Z}$ leads to a number of interesting applications, viz., networks with 'distinguished customers'; cf. [4] and [17]. Our construction for this example is carried in Section 2; it leads to a 'modified' product-form IM for the combined MP and – after the normalization – to a combined equilibrium probability distribution (EPD).

In Section 3 we discuss another instructive example: a simple exclusion model on a cubic lattice $\mathbb{Z}^d$ interacting with a special 'heavy' particle. Here again, the poduct-form for an EPD of a combined MP is preserved in the course of the cnstruction.

In Section 4 we turn to general continuous-time MPs. First, we cover a general class of jump MPs. Then a general form of our construction is provided, explaining the main mechanism behind it. In Sections 5 and 6 we focus upon examples where one or both components are diffusions. This includes queues where parameters exhibit a diffusion-type behavior as well as examples of a diffusion with a jump-like change of the drift coefficient. In particular, in Section 6 we discuss several models related to mathematical finances: here basic MPs are Ornstein–Uhlenbeck diffusion processes with varying volatility.

A number of results in this paper are stated in the form of weak invariance equations (WIEs) involving a measure and an operator acting on functions of a combined state (in a simplified form – a collection of combined jump rates). In cases where we are able to assert the existence/uniqueness of a combined MP, the WIE implies a genuine character of an IM; in these cases we refer to the invariance property directly. For convenience, the WIE is stated anew in each specific (or general) context it is used. It has to be stressed again that our results depend on a special choice of the transition mechanism (inherited from Ref. [4]) where the SE and the basic MPs have a particular

influence on each other.

Throughout the paper, we repeatedly quote books [3], [11], [13] and [14] for various general results on MPs and their generators. In fact, our models can be considered as examples of random evolution models considered in [3], Chapter 12, although we focus here on different aspects of behavior. Also, in contrast to [3], we – as was said above – include a mutual impact of the environment and basic MPs upon each other, albeit in a rather specific form. (To make a comparison: the corresponding terms used in [3] are *driving* and *driven* processes.) Technically, in the provided assertions we attempt to stick to minimal assumptions (as a rule, mentioning them in passing). But a mathematically minded reader should pay attention to remarks where we build links with more advanced conclusions by using general results from [3], [11], [13] and [14].

2 A Jackson network in a random environment

2.A. Open Jackson networks. In this section we use a Jackson job-shop network as a background model of a basic MP. The JN model is defined by the following ingredients [9, 10].

(a) A finite collection Λ of sites; a first-come-first-served infinite-buffer single-server queue at each site $j \in \Lambda$.

(b) Two vectors: $\underline{\lambda} = (\lambda_i)$, $\lambda_i \geq 0$ being a Poisson arrival intensity at site $i \in \Lambda$, and $\underline{\nu} = (\mu_k)$, $\mu_k > 0$ being the service intensity at site $k \in \Lambda$. Arrivals at different sites are independent.

(c) A sub-stochastic matrix $\mathtt{P} = (\mathrm{p}_{ik},\ i,k \in \Lambda)$: $\mathrm{p}_{ik} \geq 0$, $\sum_{k\in\Lambda} \mathrm{p}_{ik} \leq 1$. After completing service at site i, a task is transferred to site k with probability p_{ik} and leaves the network with probability $\mathrm{p}^*_i = 1 - \sum_{k\in\Lambda} \mathrm{p}_{ik}$.

The above description gives rise to a continuous-time MP with states $\underline{n} = (n_i,\ i \in \Lambda) \in \mathbb{Z}_+^\Lambda$ where $n_i \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$; this will be a prototype of the basic MP $\widetilde{X}^{(z)}(t)$. The generator matrix $\mathbf{Q} = (Q(\underline{n}, \underline{n}'))$ of the process has non-zero entries corresponding to the following transitions:

$$\begin{array}{ll} Q(\underline{n}, \underline{n} + \underline{e}^i) = \lambda_i & \text{an arrival of a task at site } i, \\ Q(\underline{n}, \underline{n} - \underline{e}^i) = \mu_i \mathrm{p}^*_i \mathbf{1}(n_i \geq 1) & \text{a task exits from site } i \text{ out of the network,} \\ Q(\underline{n}, \underline{n} + \underline{e}^{i\to k}) = \mu_i \mathrm{p}_{ik} \mathbf{1}(n_i \geq 1) & \text{a task jumps from site } i \text{ to } k. \end{array} \tag{2.1}$$

Here $\underline{e}^i = (e^i_l,\ l \in \Lambda) \in \mathbb{Z}_+^\Lambda$ has $e^i_l = \delta_{il}$, $\underline{e}^{i\to k} = \underline{e}^k - \underline{e}^i$, and $\mathbf{1}$ stands for the indicator.

(d) Let $\mathtt{I}$ denote the unit matrix. The total intensities ρ_i of the flows through sites $i \in \Lambda$ form a

vector $\underline{\rho} = (\rho_i)$ where

$$\underline{\rho} = \underline{\lambda}(\mathtt{I} - \mathtt{P})^{-1}, \text{ where } \mathtt{I} - \mathtt{P} \text{ is supposed to be invertible.} \tag{2.2}$$

(e) The IM ν for the process is given by $\nu(\underline{n}) = \prod_{i\in\Lambda} (\rho_i/\mu_i)^{n_i}$, $\underline{n} = (n_i)$.

The sub-criticality condition (SCC) reads: $\rho_i/\mu_i < 1$, $i \in \Lambda$, and gives rise to an EPD π in the form of the product of geometric marginals with parameter ρ_i/μ_i:

$$\pi(\underline{n}) = \left[\prod_{i\in\Lambda} (\rho_i/\mu_i)^{n_i}\right]/\Xi, \;\; \underline{n} = (n_i), \;\text{ where }\; \Xi = \sum_{\underline{n}\in\mathbb{Z}_+^\Lambda} \pi(\underline{n}) = \prod_{i\in\Lambda} (1 - \rho_i/\mu_i). \tag{2.3}$$

In fact, assuming that matrix $\mathtt{P}$ is irreducible (i.e., $\mathtt{P}^s$ has strictly positive entries for some positive integer s) and that the SCC holds, the JN MP is positive recurrent, and π is the unique EPD.

2.B. The combined generator. We now give the description of the model modifying the basic example from [4]. Let $\mathcal{Z}$ be a finite or a countable set. The state of the combined MP is a pair $(z, \underline{n})$ where $z \in \mathcal{Z}$ indicates a state of the environment (SE) and $\underline{n} \in \mathbb{Z}^\Lambda$ a state of a basic process. Furthermore, given $z \in \mathcal{Z}$, we fix a collection of triples $(\underline{\lambda}^{(z)}, \underline{\nu}^{(z)}, \mathtt{P}^{(z)})$, with $\underline{\lambda}^{(z)} = (\lambda_i^{(z)})$, $\underline{\mu}^{(z)} = (\mu_i^{(z)})$, $\mathtt{P}^{(z)} = (\mathrm{p}_{ik}^{(z)})$, assuming that, $\forall\; z \in \mathcal{Z}$, Eqns (2.1)–(2.2) hold, and matrix $\mathtt{P}^{(z)}$ is irreducible. The jump rates form an (infinite) generator matrix $\mathbf{R} = \Big(R[(z,\underline{n}),(z',\underline{n}')]\Big)$. The entries $R[(z,\underline{n}),(z',\underline{n}')]$ are:

$$\begin{array}{ll}
R[(z,\underline{n}),(z,\underline{n}+\underline{e}^i)] = \alpha(z)\lambda_i^{(z)} & \text{a task arrival at site } i \text{ under SE } z,\\
R[(z,\underline{n}),(z,\underline{n}-\underline{e}^i)] = \alpha(z)\mu_i^{(z)}\mathrm{p}_i^{*(z)}\mathbf{1}(n_i \geq 1) & \text{a task exit from site } i \text{ under SE } z,\\
R[(z,\underline{n}),(z,\underline{n}+\underline{e}^{i\to k})] = \alpha(z)\mu_i^{(z)}\mathrm{p}_{ik}^{(z)}\mathbf{1}(n_i \geq 1) & \text{a task moves } i \to k \text{ under SE } z,\\
R[(z,\underline{n}),(z',\underline{n})] = \sigma(z)\tau^{(\underline{n})}(z,z')\prod\limits_{i\in\Lambda}\left(\rho_i^{(z)}/\mu_i^{(z)}\right)^{-n_i} & \text{the SE changes from } z \text{ to } z',\\
R[(z,\underline{n}),(z',\underline{n}')] = 0 & \forall \text{ other pair } (z',\underline{n}') \in \mathcal{Z}\times\mathbb{Z}_+^\Lambda\\
 & \text{with } (z,\underline{n}) \neq (z',\underline{n}'),\\
R[(z,\underline{n}),(z,\underline{n})] = -\sum\limits_{(z',\underline{n}')\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda\setminus\{(z,\underline{n})\}} R[(z,\underline{n}),(z',\underline{n}')]. &
\end{array} \tag{2.4}$$

Here $(\alpha(z),\; z \in \mathcal{Z}) \in \mathbb{R}_+^{\mathcal{Z}}$ and $(\sigma(z),\; z \in \mathcal{Z}) \in \mathbb{R}_+^{\mathcal{Z}}$ are given vectors with entries $\alpha(z) \in [0,\infty)$, $\sigma(z) \in (0,\infty)$; they represent two forms of time-scaling: for the basic MP under SE z and for the exit from z. Next, $\mathrm{T}^{(\underline{n})} = (\tau^{(\underline{n})}(z,z'))$ is the nominal SE-jump intensity matrix (possibly, depending on $\underline{n}$) for which we assume that

$$\tau^{(\underline{n})}(z,z') \geq 0, \;\; \sum_{z'\in\mathcal{Z}} \tau^{(\underline{n})}(z,z') = \sum_{z'\in\mathcal{Z}} \tau^{(\underline{n})}(z',z), \;\; \tau(z,z) = 0, \;\; \forall\;\; z \in \mathcal{Z},\; \underline{n} \in \mathbb{Z}_+^\Lambda. \tag{2.5}$$

One can see that the top three lines in (2.4) emerge from the jump rates of a basic MP $\widetilde{X}^{(z)}(t)$ whereas the bottom line is related to the SE MP $\widetilde{Z}(t)$.

We say that measure $\boldsymbol{\eta}$ on $\mathcal{Z}\times\mathbb{Z}_+^\Lambda$ and the collection of rates $\mathbf{R}=\{R[(z,\underline{n}),(z',\underline{n}')]\}$ satisfy the WIE (the weak invariance equation) if, $\forall\ (z,\underline{n})\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda$,

$$\sum_{(z',\underline{n}')\neq(z,\underline{n})}\boldsymbol{\eta}(z,\underline{n})R[(z,\underline{n}),(z',\underline{n}')]=\sum_{(z',\underline{n}')\neq(z,\underline{n})}\boldsymbol{\eta}(z',\underline{n}')R[(z',\underline{n}'),(z,\underline{n})]. \tag{2.6}$$

Theorem 2.1. (I) *Set:*

$$\boldsymbol{\kappa}(z,\underline{n})=\left[\prod\nolimits_{i\in\Lambda}\left(\rho_i^{(z)}/\mu_i^{(z)}\right)^{n_i}\right]/\sigma(z),\quad z\in\mathcal{Z},\ \underline{n}=(n_i)\in\mathbb{Z}^\Lambda. \tag{2.7}$$

Then measure $\boldsymbol{\kappa}$ and the collection of rates from Eqn (2.4) *satisfy the WIE.*

(II) *Assuming the SCC*

$$\rho_i^{(z)}/\mu_i^{(z)}<1\ \ \forall\ \ i,z\ \ \text{and}\ \ \ \Xi=\sum\nolimits_{z\in\mathcal{Z}}\left[\prod\nolimits_{i\in\Lambda}\left(1-\rho_i^{(z)}/\mu_i^{(z)}\right)\right]/\sigma(z)<\infty \tag{2.8}$$

yields a PM $\boldsymbol{\pi}$ satisfying the WIE: $\boldsymbol{\pi}(z,\underline{n})=\left[\prod\nolimits_{i\in\Lambda}\left(\rho_i^{(z)}/\mu_i^{(z)}\right)^{n_i}\right]/[\Xi(\Lambda)\sigma(z)]$.

Proof. Assertion (II) follows from (I) so we focus on the proof of (I). We check partial balance equations: $\forall\ (z,\underline{n})$, we assert that $F^{\rm out}(z,\underline{n})=F^{\rm in}(z,\underline{n})$. Here

$$F^{\rm out}(z,\underline{n})=\sum\nolimits_{(z',\underline{n}')}\boldsymbol{\kappa}(z,\underline{n})R[(z,\underline{n}),(z',\underline{n}')],\ F^{\rm in}(z,\underline{n})=\sum\nolimits_{(z',\underline{n}')}\boldsymbol{\kappa}(z',\underline{n}')R[(z',\underline{n}'),(z,\underline{n})].$$

It is convenient to represent $F^{\rm out}(z,\underline{n})=F_1^{\rm out}(z,\underline{n})+F_2^{\rm out}(z,\underline{n})$, $F^{\rm in}(z,\underline{n})=F_1^{\rm in}(z,\underline{n})+F_2^{\rm in}(z,\underline{n})$, and prove that $F_1^{\rm out}(z,\underline{n})=F_1^{\rm in}(z,\underline{n})$, $\ F_2^{\rm out}(z,\underline{n})=F_2^{\rm in}(z,\underline{n})$, with

$$F_1^{\rm out}(z,\underline{n})=\sum\nolimits_{\underline{n}'}\boldsymbol{\kappa}(z,\underline{n})R[(z,\underline{n}),(z,\underline{n}')],\ F_2^{\rm out}(z,\underline{n})=\sum\nolimits_{z'}\boldsymbol{\kappa}(z,\underline{n})R[(z,\underline{n}),(z',\underline{n})],$$

and

$$F_1^{\rm in}(z,\underline{n})=\sum\nolimits_{\underline{n}'}\boldsymbol{\kappa}(z,\underline{n}')R[(z,\underline{n}'),(z,\underline{n})],\ F_2^{\rm in}(z,\underline{n})=\sum\nolimits_{z'}\boldsymbol{\kappa}(z',\underline{n})R[(z',\underline{n}),(z,\underline{n})].$$

After omitting the factor $1/\sigma(z)$, the equation $F_1^{\rm out}(z,\underline{n})=F_1^{\rm in}(z,\underline{n})$ means that

$$\sum\nolimits_{\underline{n}'}\nu^{(z)}(\underline{n})Q^{(z)}(\underline{n},\underline{n}')=\sum\nolimits_{\underline{n}'}\nu^{(z)}(\underline{n}')Q^{(z)}(\underline{n}',\underline{n})$$

which holds as $\nu^{(z)}$ is an IM for $Q^{(z)}$. Next, $F_2^{\rm out}(z,\underline{n})=F_2^{\rm in}(z,\underline{n})$ is equivalent to (2.5) since

$$F_2^{\rm out}(z,\underline{n})=\sum\nolimits_{z'}\left(1/\sigma(z)\right)\prod\nolimits_i\left(\rho_i^{(z)}/\mu_i^{(z)}\right)^{n_i}\tau^{(\underline{n})}(z,z')\prod\nolimits_i\left(\rho_i^{(z)}/\mu_i^{(z)}\right)^{-n_i}\sigma(z)=\sum\nolimits_{z'}\tau^{(\underline{n})}(z,z')$$

and similarly $F_2^{\rm in}(z,\underline{n})=\sum\nolimits_{z'}\tau^{(\underline{n})}(z',z)$. $\square$

Remarks. 2.1. Note that values $\alpha(z)$ and $\tau^{(\underline{n})}(z,z')$ do not enter expression (2.7). However, if $\alpha(z)\equiv 0$ then $\forall$ fixed $\underline{n}$ pairs $(z,\underline{n})$, $z\in\mathcal{Z}$, form a closed communicating class supporting an IM $\boldsymbol{\kappa}_0=\boldsymbol{\kappa}_0^{(\underline{n})}$ with values $\boldsymbol{\kappa}_0(z,\underline{n})=1/\sigma(z)$.

2.2. Under additional conditions of a 'moderate growth of functions $\alpha(z)$ and $\sigma(z)$ (cf. Theorem 3.1, P. 376 and Corollary 3.2, P. 379 in [3]), the collection of rates (2.4) defines a combined MP $(Z(t), X(t))$ on $\mathcal{Z} \times \mathbb{Z}_+^\Lambda$ with a Feller semi-group of transition operators. (Here it means that, $\forall$ $t > 0$, the transition matrix of the process takes the space of bounded functions $(z, \underline{n}) \mapsto \phi(z, \underline{n})$ to itself). Physically speaking, for 'nice' $\alpha(z)$ and $\sigma(z)$ we obtain a non-explosive combined MP. (For a finite set $\mathcal{Z}$ this is automatically true.)

Formally, a sufficient condition (deduced from the aforementioned results in [3]) is that the following three assumptions (i)–(iii) hold. (i) For $\overline{R}(z, \underline{n}) = -R[(z, \underline{n}), (z, \underline{n})]$ we assume:

$$\sup_{(z,\underline{n})\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda} \overline{R}(z, \underline{n}) < \infty \ \text{ where}$$

$$\overline{R}(z, \underline{n}) = \sigma(z) \sum_{z'\in\mathcal{Z}} \tau_{zz'}^{(\underline{n})} \prod_{i\in\Lambda} \left(\frac{\rho_i^{(z)}}{\mu_i^{(z)}} \right)^{-n_i} + \alpha(z) \sum_{i\in\Lambda} \big[\lambda_i^{(z)} + \mu_i^{(z)} \mathbf{1}(n_i \geq 1) \big]. \tag{2.9}$$

Next, (ii) after compactifying space $\mathcal{Z}\times\mathbb{Z}_+^\Lambda$ by a point Δ, we suppose that $\forall$ finite subset $\mathcal{K} \in \mathcal{Z}\times\mathbb{Z}_+^\Lambda$,

$$\lim_{(z,\underline{n})\to\Delta} \sum_{(z',\underline{n}')\in\mathcal{K}} \big| R[(z, \underline{n}), (z', \underline{n}')] \big| = 0, \tag{2.10}$$

Finally, (iii) we assume that

$$\sup_{(z,\underline{n})\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda} \sum_{(z',\underline{n}')\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda} \left| \Big\{ \overline{R}(z, \underline{n}) - \overline{R}(z', \underline{n}') \Big\} R[(z, \underline{n}), (z', \underline{n}')] \right| < \infty. \tag{2.11}$$

Results about stationary distributions (see Proposition 9.2, P. 239, from [3]) imply that under assumptions (i)–(iii), a measure satisfying the WIE is invariant under the transition semi-group. Furthermore, under assumptions (i) – (iii) and condition (2.8), the combined MP $(Z(t), X(t))$ is positive recurrent.

2.3. In assumption (i) (cf. Eqn (2.9)), the troublesome summand is

$$\overline{R}^{(1)}(z, \underline{n}) := \sigma(z) \sum_{z'\in\mathcal{Z}} \tau_{zz'}^{(\underline{n})} \prod_{i\in\Lambda} \left(\frac{\mu_i^{(z)}}{\rho_i^{(z)}} \right)^{n_i}.$$

A sufficient condition for $\sup\limits_{(z,\underline{n})\in\mathcal{Z}\times\mathbb{Z}_+^\Lambda} \overline{R}^{(1)}(z, \underline{n}) < \infty$ covering a host of realistic situations is that (iv) $\tau_{zz'}^{(\underline{n})}$ is of the form $\tau_{zz'}^{(\underline{n})} = h(\underline{n})\overline{\tau}_{zz'}$ where $\overline{\tau}_{zz'}$ does not depend on $\underline{n}$ and has $\sup\limits_{z\in\mathcal{Z}} \sum\limits_{z'\in\mathcal{Z}} \overline{\tau}_{zz'} < \infty$ and (v) $h(\underline{n})$ is such that $\sup\limits_{z\in\mathcal{Z},\underline{n}\in\mathbb{Z}_+^\Lambda} h(\underline{n}) \prod\limits_{i\in\Lambda} \left(\frac{\mu_i^{(z)}}{\rho_i^{(z)}} \right)^{n_i} < \infty$. For example, assuming that $S(n) := \sup\limits_{z\in\mathcal{Z}} \prod\limits_{i\in\Lambda} \left(\frac{\mu_i^{(z)}}{\rho_i^{(z)}} \right)^{n_i} < \infty$, the value $h(\underline{n})$ can be selected as $S(n)^{-1}$.

3 A simple exclusion model in a random environment

In this section, we focus on a class of MPs with local interaction arising from a symmetric simple exclusion model; cf.[13]–[14]. The state x of the basic MP is considered as a spin-$0,1$ configuration (aka an occupancy configuration) on a cubic lattice $\mathbb{Z}^d$, i.e., as a function $i \in \mathbb{Z}^d \mapsto \mathrm{x}_i \in \{0,1\}$. The value $\mathrm{x}_i = 1$ means that site i is occupied by a 'particle' whereas $\mathrm{x}_i = 0$ means that site i is vacant/empty. (We also will use the term a 'light particle' as opposite to a 'heavy' particle introduced below.) Let us fix parameters

$$\begin{array}{c}\varphi \in \mathbb{R},\ \lambda, \mu > 0,\ \beta_{i,i'} = \beta_{i'i}\\ \text{where } i, i' \in \mathbb{Z}^d,\ \beta_{ii} = 0 \text{ and } \sup\limits_{i\in\mathbb{Z}^d} \sum\limits_{i'\in\mathbb{Z}^d} \beta_{ii'} < \infty.\end{array} \tag{3.1}$$

Next, we take $\mathbb{X} = \{0,1\}^{\mathbb{Z}^d}$ with the product topology (and – when necessary – with a metric generating this topology), and set $\mathcal{Z} = \mathbb{T}$ where $\mathbb{T} \subset \mathbb{Z}^d$ is a finite set. Further, for $x = (\mathrm{x}_i,\ i \in \mathbb{Z}^d) \in \{0,1\}^{\mathbb{Z}^d}$ and $z \in \mathbb{T}$ we write:

$$\begin{array}{l}\mathtt{L}^{(z)} g(x) = \sum\limits_{i,i'\in\mathbb{Z}^d:\, i\neq i'\neq z} \mathrm{x}_1(1-\mathrm{x}_{i'})\beta_{i,i'}\big[g(x+e^{i\to i'}) - g(x)\big] \qquad | \\ \quad + \sum\limits_{i\in\mathbb{Z}^d:\, i\neq z} \Big\{\mathrm{x}_i(1-\mathrm{x}_i)\theta_{iz}e^{\varphi}\big[g(x+e^{i\to z}) - g(x)\big] \\ \qquad +\lambda\big[g(x+e^i) - g(x)\big] + \mathrm{x}_i\mu\big[g(x-e^i) - g(x)\big]\Big\} \\ \quad +\lambda e^{\varphi}\big[g(x+e^z) - g(x)\big] + \mathrm{x}_z\mu\big[g(x-e^z) - g(x)\big].\end{array} \tag{3.2}$$

Here $e^i \in \{0,1\}^{\mathbb{Z}^d}$ is the configuration where all values of spin are 0 except for that at site i, which is 1, and we set: $e^{i\to i'} := e^i - e^{i'}$. (This covers the case where i or i' coincides with z.) Adding and subtracting configurations means addition and subtraction of functions.

A construction below repeats that from [13], Sect. I.3. Denote by $\mathrm{C_L}(\{0,1\}^{\mathbb{Z}^d})$ the space of continuous functions $\mathrm{C}(\{0,1\} \times \mathbb{Z}^d)$. Operator $\mathtt{L}^{(z)}$ in (3.2) acts initially on the space $\mathrm{C_L}(\{0,1\}^{\mathbb{Z}^d})$ of Lipschitz-type functions $x \in \{0,1\}^{\mathbb{Z}^d} \mapsto g(x)$:

$$\begin{array}{l}\mathrm{C_L}(\{0,1\}^{\mathbb{Z}^d}) = \Big\{g \in \mathrm{C}(\{0,1\}^{\mathbb{Z}^d}):\ |||g||| := \sup\limits_{x\in\{0,1\}^{\mathbb{Z}^d}} \Big[\sum\limits_{i\in\mathbb{Z}^d} \mathrm{x}_i \left|g\left(x-e^i\right) - g(x)\right| \\ \quad + \sum\limits_{i\in\mathbb{Z}^d} (1-\mathrm{x}_i)\left|g\left(x+e^i\right) - g(x)\right| + \sum\limits_{i,i'\in\mathbb{Z}^d} \mathrm{x}_i(1-\mathrm{x}_{i'})\left|g\left(x+e^{i\to i'}\right) - g(x)\right|\Big] < \infty\Big\}.\end{array} \tag{3.3}$$

Next, we extend (3.2) to a closed operator in $\mathrm{C}(\{0,1\}^{\mathbb{Z}^d})$. The closed operator is still denoted by $\mathtt{L}^{(z)}$; for its domain we use the notation $\mathrm{D}(\mathtt{L}^{(z)})$. By construction, $\mathrm{C_L}(\{0,1\}^{\mathbb{Z}^d})$ is a core for $\mathtt{L}^{(z)}$. Physically, $\mathtt{L}^{(z)}$ is a generator of an MP $\widetilde{X}^{(z)}$ on the state space $\{0,1\}^{\mathbb{Z}^d}$ representing an 'open' simple-exclusion model, in presence of a (single) 'heavy particle' placed at site $z \in \mathbb{T}$. In this model, a light particle

can jump from site i to site i' at rates $\beta_{ii'}$ or $\beta_{ii'}e^{\varphi}$, depending on the status of sites i and i' (vacant, occupied by a light particle, occupied/not occupied by a heavy particle). In any case, jumps are performed only if the simple-exclusion restriction is respected: at most one light particle at any given site. (A simultaneous presence of a light and the heavy particle at a given site is allowed.) A light particle can also be annihilated at rate μ and created at rates λ or λe^{ϕ}, depending on whetger the given site is occupied by the heavy particle. Factor e^{ϕ} indicates an impact that a heavy particle has on the dynamics of a light-particle configuration x: when $\varphi > 0$, the heavy particle attracts the light ones, when $\varphi < 0$, it repels them. Following [13]–[14], it is possible to check that $\forall\ z \in \mathbb{T}$ there exists a unique Feller semi-group of operators in $\mathrm{C}(\{0,1\}^{\mathbb{Z}^d})$ generated by $\mathtt{L}^{(z)}$. See, e.g., [13], Theorem I.3.9, P. 27, or [14], Theorem 4.68. This yields a basic MP $\widetilde{X}^{(z)}$ in $\{0,1\}^{\mathbb{Z}^d}$.

Furthermore, an IM $\nu^{(z)}$ (in fact, an EPD) for process $\widetilde{X}^{(z)}$ is given by

$$\nu^{(z)} = \left(\prod_{i\in\mathbb{Z}^d:\, i\neq z} \mathtt{P}^{(i)}\right) \times \mathtt{Q}^{(z)}. \tag{3.4}$$

It is a product-measure (aka an inhomogeneous Bernoulli measure) where the spin values are independent for different sites, and the marginal distribution for an individual spin is either $\mathtt{P}^{(i)} \simeq \mathtt{P}$ or $\mathtt{Q}^{(i)} \simeq \mathtt{Q}$, depending on whether $i \neq z$ (i.e., site i not occupied by a heavy particle) or $i = z$ (i.e., i contains the heavy particle). Both $\mathtt{P}$ and $\mathtt{Q}$ are probability distributions on the two-outcome set $\{0,1\}$:

$$\mathtt{P}(1) = \frac{\lambda}{\lambda+\mu},\ \mathtt{P}(0) = \frac{\mu}{\lambda+\mu},\ \mathtt{Q}(1) = \frac{\lambda e^{\varphi}}{\lambda e^{\varphi}+\mu},\ \ \mathtt{Q}(0) = \frac{\mu}{\lambda e^{\varphi}+\mu}. \tag{3.5}$$

All measures $\nu^{(z)}$ are absolutely continuous relative to $\gamma = \underset{i\in\mathbb{Z}^d}{\times} \mathtt{P}^{(i)}$, with the Radon–Nikodym densities

$$m(z,x) = \frac{\mathrm{d}\nu^{(z)}(x)}{\mathrm{d}\gamma(x)} = \frac{\lambda+\mu}{\lambda e^{\varphi}+\mu}\Big[e^{\varphi}\mathbf{1}(\mathrm{x}_z = 1) + \mathbf{1}(\mathrm{x}_z = 0)\Big] = \frac{\lambda+\mu}{\lambda e^{\varphi}+\mu}e^{\mathrm{x}_z\phi} \\ \text{where } z\in\mathbb{T} \ \text{ and } \ x = (\mathrm{x}_i) \in \{0,1\}^{\mathbb{Z}^d}. \tag{3.6}$$

We also have that $\int_{\{0,1\}^{\mathbb{Z}^d}} \mathtt{L}^{(z)} g(x)\mathrm{d}\nu^{(z)}(x) = 0$, $\forall\ z \in \mathbb{T}$ and $g \in \mathrm{C}_{\mathrm{L}}(\{0,1\}^{\mathbb{Z}^d})$.

Next, the SE (state of environment) process $\widetilde{Z}(t)$ has the generator $\mathtt{A}$ defined by a finite matrix for a function/vector $z \in \mathbb{T} \mapsto f(z)$,

$$\mathtt{A}f(z) = \sum_{z'\in\mathbb{T}} \tau_{zz'}[f(z') - f(z)] \text{ where rates } \ \tau_{zz'} \ \text{ obey} \\ \tau_{zz'} = \tau_{z'z} \geq 0, \ \text{ and } \ \tau_{zz} = 0. \tag{3.7}$$

Process $\widetilde{Z}(t)$ is a random walk in $\mathbb{T}$ with a counting invariant measure γ: $\gamma(\mathbb{D}) = \#\mathbb{D}$, $\mathbb{D} \subseteq \mathbb{T}$. The invariance property simply means that $\sum_{z'} \tau_{zz'} = \sum_{z'} \tau_{z'z}$ and is deduced from the symmetry condition $\tau_{zz'} = \tau_{z'z}$ in (3.7). It implies that $\sum_{z\in\mathbb{T}} \mathtt{A}f(z) = 0$ for any function f.

Further, consider a measure $\boldsymbol{\kappa}$ on $\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d}$:

$$\boldsymbol{\kappa}(z, \mathbb{A}) = \frac{1}{\sigma(z)} \int_{\mathbb{A}} e^{\mathrm{x}_z \varphi} \mathrm{d}\gamma(x), \quad z \in \mathbb{T}, \quad \mathbb{A} \subseteq \{0,1\}^{\mathbb{Z}^d}. \tag{3.8}$$

The combined generator $\mathbf{R}$ is constructed from the action

$$\mathbf{R}\phi(z,x) = \alpha(z)\mathbf{L}^{(z)}\phi(z,x) + \frac{\sigma(z)}{e^{\mathrm{x}_z\varphi}}\mathbf{A}\phi(z,x). \tag{3.9}$$

Here $\alpha(z) > 0$ and $\sigma(z) > 0$ are time-scaling coefficients. Next, $\mathbf{L}^{(z)}$ acts on the section map $x \in \{0,1\}^{\mathbb{Z}^d} \mapsto \phi(z,x)$:

$$\begin{aligned}
\mathbf{L}^{(z)}\phi(z,x) = &\sum_{i,i' \in \mathbb{Z}^d:\, i \neq i' \neq z} \mathrm{x}_i(1-\mathrm{x}_{i'})\beta_{i,i'}\Big[\phi(z, x + e^{i\to i'}) - \phi(z,x)\Big] \\
&+ \sum_{i \in \mathbb{Z}^d:\, i \neq z} \Big\{ \mathrm{x}_i(1-\mathrm{x}_z)\beta_{iz}e^{\varphi}\big[\phi(z, x+e^{i\to z}) - \phi(z,x)\big]\Big\} \\
&\quad + \lambda\big[\phi(z, x+e^i) - \phi(z,x)\big] + \mathrm{x}_i\mu\big[\phi(z, x-e^i) - \phi(z,x)\big]\Big\} \\
&+ \lambda e^{\varphi}\big[\phi(z, x+e^z) - \phi(z,x)\big] + \mathrm{x}_z\mu\big[\phi(z, x-e^z) - \phi(z,x)\big].
\end{aligned} \tag{3.10}$$

Further,

$$\mathbf{A}\phi(z,x) = \sum_{z' \in \mathbb{Z}^d} \tau_{zz'}[\phi(z', x) - \phi(z,x)]. \tag{3.11}$$

(The constant factor $\dfrac{\lambda e^{\varphi} + \mu}{\lambda + \mu}$ has been absorbed into $\sigma(z)$.)

Let $\mathrm{C}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d}) = \mathrm{C}(\{0,1\}^{\mathbb{Z}^d})^{\mathbb{T}}$ be the space of continuous functions $\mathrm{C}_{\mathrm{L}}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d})$ on $\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d}$. As in the case of $\mathtt{L}^{(x)}$, we consider $\mathbf{R}$ initially on space $\mathrm{C}_{\mathrm{L}}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d})$ of Lipschitz-type functions:

$$\begin{aligned}
\mathrm{C}_{\mathrm{L}}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d}) = \Big\{ &\phi \in \mathrm{C}(\{0,1\}^{\mathbb{Z}^d}) : \ \forall\, z \in \mathbb{T}, \\
&\text{the section map } x \in \{0,1\}^{\mathbb{Z}^d} \mapsto \phi(z,x) \text{ lies in } \mathrm{C}_{\mathrm{L}}(\{0,1\}^{\mathbb{Z}^d}); \text{ cf. (3.3)} \Big\}.
\end{aligned} \tag{3.12}$$

Then take the closure in $\mathrm{C}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d})$, keeping for the obtained closed operator the same notation $\mathbf{R}$. By construction, $\mathrm{C}_{\mathrm{L}}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d})$ is a core for $\mathbf{R}$. Owing to Theorem 3.9 from [13] and/or Theorem 4.68 from [14], there exists a unique Feller semi-group on $\mathrm{C}(\mathbb{T} \times \{0,1\}^{\mathbb{Z}^d})$ generated by $\mathbf{R}$; the corresponding MP is denoted by $(Z(t), X(t))$. The trajectories of process $(Z(t), X(t))$ are right-continuous maps $[0,\infty) \mapsto \mathbb{T} \times \{0,1\}^{\mathbb{Z}^d}$ with left limits.

Theorem 3.1. *Under the above conditions, $\boldsymbol{\kappa}$ is a finite IM for process $(Z(t), X(t))$.*

Proof. By virtue of Proposition I.6.10 in [13], P. 52, it suffices to check the equation $\int_{\mathbb{T}\times\{0,1\}^{\mathbb{Z}^d}}(\mathbf{R}\phi)(z,x)\mathrm{d}\boldsymbol{\kappa}(z,x)=0$ for $\phi\in\mathrm{C_L}(\mathbb{T}\times\{0,1\}^{\mathbb{Z}^d})$. But for any such function ϕ,

$$\begin{aligned}&\int_{\mathbb{T}\times\{0,1\}^{\mathbb{Z}^d}}(\mathbf{R}\phi)(z,x)\mathrm{d}\boldsymbol{\kappa}(z,x)\\&\quad=\sum_{z\in\mathbb{T}}\frac{\alpha(z)}{\sigma(z)}\int_{\{0,1\}^{\mathbb{Z}^d}}(\mathbf{L}^{(z)}\phi)(z,x)e^{\mathrm{x}_z\varphi}\mathrm{d}\gamma(x)+\int_{\{0,1\}^{\mathbb{Z}^d}}\sum_{z\in\mathbb{T}}(\mathbf{A}\phi)(z,x)\mathrm{d}\gamma(x).\end{aligned}$$

Observe that $\forall\ z\in\mathbb{T}$, owing to the fact that the section map of ϕ belongs to $\mathrm{C_L}(\{0,1\}^{\mathbb{Z}^d})$,

$$\int_{\{0,1\}^{\mathbb{Z}^d}}(\mathbf{L}^{(z)}\phi)(z,x)e^{\mathrm{x}_z\varphi}\mathrm{d}\gamma(x)=\frac{\lambda+\mu}{\lambda e^{\varphi}+\mu}\int_{\{0,1\}^{\mathbb{Z}^d}}(\mathbf{L}^{(z)}\phi)(z,x)\mathrm{d}\nu^{(z)}(x)=0.$$

Also, for γ-a.a. $x\in\{0,1\}^{\mathbb{Z}^d}$, we have that $\sum\limits_{z\in\mathbb{T}}(\mathbf{A}\phi)(z,x)=0$. Hence, $\int_{\mathbb{T}\times\{0,1\}^{\mathbb{Z}^d}}(\mathbf{R}\phi)\mathrm{d}\boldsymbol{\kappa}=0$.

4 A general construction

4.A. Jump Markov processes. Theorem 1 admits an extention where $(\mathcal{Z},\upsilon)$ is a (standard) measure space. (Usual measurability and countable additivity assumptions must be adopted throughout this sub-section.) Here, sums $\sum_z$ and $\sum_{z'}$ are replaced by integrals against measure υ; measures on space $\mathcal{Z}\times\mathbb{Z}^\Lambda$ are supposed to be given via (Radon–Nikodym) densities wrt υ times the counting measure. Viz. in (2.8) one reads $\boldsymbol{\Xi}=\int_{\mathcal{Z}}\left\{\left[\prod_{i\in\Lambda}\left(1-\rho_i^{(z)}/\mu_i^{(z)}\right)\right]/\sigma(z)\right\}\mathrm{d}\upsilon(z)<\infty$. Existence of the combined MP $(Z(t),X(t))$ should be analysed separately; cf. Remarcs 2B, 2C.

Next, $\mathbb{Z}^\Lambda$ can also be replaced with a measure space, $(\mathbb{X},\gamma)$; here the combined state space is again the Cartesian product $(\mathcal{Z}\times\mathbb{X},\upsilon\times\gamma)$. We refer the reader to in [3], Section 2 of Chapter 4, PP. 162–173, and Section 3 of Chapter 8, PP. 376–382 , for a detailed treatment of jump MPs and their generators. Assume that for $x\in\mathbb{X}$ we have a kernel $\tau^{(x)}(z,z')\geq 0$, $z,z'\in\mathcal{Z}$, and for $z\in\mathcal{Z}$ a kernel $Q^{(z)}(x,x')\geq 0$ and a function $m(z,x)>0$, $x,x'\in\mathbb{X}$, such that $\tau^{(x)}(z,z)=Q^{(z)}(x,x)=0$ and

$$\begin{array}{r}\int_{\mathcal{Z}}\tau^{(x)}(z,z')\mathrm{d}\upsilon(z')=\int_{\mathcal{Z}}\tau^{(x)}(z',z)\mathrm{d}\upsilon(z'),\\ m(z,x)\int_{\mathbb{X}}Q^{(z)}(x,y)\mathrm{d}\gamma(y)=\int_{\mathbb{X}}m(z,y)Q^{(z)}(y,x)\mathrm{d}\gamma(y),\end{array}\quad\text{for }(\upsilon\times\gamma)\text{-a.a }(z,x)\in\mathcal{Z}\times\mathbb{X}.\tag{4.1}$$

Physically speaking, Eqn (4.1) yields that the environment MPs have an IM υ whereas the basic MP under SE (state of environment) z has an IM $\nu^{(z)}$ with Radon–Nikodym density $\dfrac{\nu^{(z)}(\mathrm{d}x)}{\gamma(\mathrm{d}x)}=m(z,x)$. (As above, a formal construction of these MPs requires additional assumptions discussed in [3].)

The combined generator has rates $R[(z,x),(z',x')]$:

$$R[(z,x),(z,x')]=\alpha(z)Q^{(z)}(x,x'),\quad R[(z,x),(z',x)]=\sigma(z)\frac{\tau^{(x)}(z,z')}{m(z,x)},\tag{4.2}$$

with time-scale coefficients $\alpha(z) \geq 0$, $\sigma(z) > 0$ having the same meaning as before. We say that a density $\boldsymbol{\eta}(z,x)$ on $(\mathcal{Z}\times\mathbb{X}, \upsilon\times\gamma)$ and kernel $\mathbf{R} = \{R[(z,x),(z',x')]\}$ satisfy the WIE if, for $(\upsilon\times\gamma)$-a.a $(z,x) \in \mathcal{Z}\times\mathbb{X}$,

$$\begin{aligned}\boldsymbol{\eta}(z,x)\int_{\mathcal{Z}\times\mathbb{X}} R[(z,x)(z',x')]\mathbf{1}((z,x)\neq(z',x'))\mathrm{d}\upsilon(z')\mathrm{d}\gamma(x')\\ = \int_{\mathcal{Z}\times\mathbb{X}} \boldsymbol{\eta}(z',x'))R[(z',x'),(z,x)]\mathbf{1}((z,x)\neq(z',x'))\mathrm{d}\upsilon(z')\mathrm{d}\gamma(x').\end{aligned} \tag{4.3}$$

Theorem 4.1. (I) *The density* $\boldsymbol{\kappa}(z,x) = \nu^{(z)}(x)/\sigma(z)$ *considered relative to measure* $\upsilon\times\gamma$ *and the kernel* $R[(z,x),(z',x')]$ *from Eqn* (4.2) *satisfy the WIE on* $\mathcal{Z}\times\mathbb{X}$. (II) *Under the SCC* $\Xi = \int_{\mathcal{Z}\times\mathbb{X}} \boldsymbol{\kappa}(z,x)\mathrm{d}\upsilon(z)\mathrm{d}\gamma(x) < \infty$, *the PD* $\boldsymbol{\pi}(z,x) = \boldsymbol{\kappa}(z,x)/\Xi$ *also satisfies the WIE.*

Proof: Repeats that of Theorem 2.1 by replacing sums with integrals. $\square$

Remarks. 4.1. As before, the kernels $\tau^{(x)}(z,z')$ enter the density $\boldsymbol{\kappa}$ indirectly, through measure υ and condition (4.1).

4.2. Under additional assumptions on spaces $(\mathcal{Z},\upsilon)$ and $(\mathbb{X},\gamma)$, functions $\alpha(z)$ and $\sigma(z)$ and kernels $\tau^{(x)}(z,z')$ and $Q^{(z)}(x,x')$ (see Theorem 3.1, P. 377 in [3]), operator $\mathbf{R}$ generates a strongly continuous Feller semi-group on $\mathcal{Z}\times\mathbb{X}$. It can also be achieved that $\boldsymbol{\kappa}$ yields an IM for this corresponding MP.

4.B. General assumptions on component MPs. In sub-sections 4.B – 4.D we develop further a general construction of a combined generator. Until the end of Section 4, the spaces $\mathcal{Z}$ and $\mathbb{X}$ below are supposed to be locally compact Polish, and all measures under consideration are Borel, countably-additive and finite on compact sets.

Let us proceed with formal definitions. As before, an SE is represented by a point $z \in \mathcal{Z}$. We also assume that we have a measure υ on $\mathcal{Z}$ which will serve as an IM for environment MPs.

Next, there is a space $\mathbb{X}$ given; points $x \in \mathbb{X}$ are treated as states of a family of MPs indexed by $z \in \mathcal{Z}$ – we again call them basic MPs. Physically speaking, each basic MP has a generator and an IM indexed by $z \in \mathcal{Z}$. More precisely, $\forall\ z \in \mathcal{Z}$, we have a measure $\nu^{(z)}$ on $\mathbb{X}$; we suppose that each measure $\nu^{(z)}$ is absolutely continuous wrt a fixed measure γ, with Radon–Nikodym densities $m(z,x) = \dfrac{\mathrm{d}\nu^{(z)}(x)}{\mathrm{d}\gamma(x)} > 0$ for $(\upsilon\times\gamma)$-a.a. $(z,x) \in \mathcal{Z}\times\mathbb{X}$.

Further, $\forall\ z \in \mathcal{Z}$ a closed linear operator $\mathtt{L}^{(z)}$ is given, acting on functions $g : \mathbb{X} \to \mathbb{R}$ forming a domain $\mathrm{D}(\mathtt{L}^{(z)})$ dense in $\mathrm{C_b}(\mathbb{X})$, the space of bounded continuous functions on $\mathbb{X}$. Further still, we suppose that each $\mathtt{L}^{(z)}$ is a generator of a Feller MP on $\mathbb{X}$. That is, $\mathtt{L}^{(z)}$ is a conservative operator satisfying the assumptions of Theorem 2.2, P. 165 in [3] (a version of the Hille–Yosida theorem). We also assume that $\forall\ g$ from a core of $\mathtt{L}^{(z)}$, the integral $\int_{\mathbb{X}}(\mathtt{L}^{(z)}g)(x)\mathrm{d}\nu^{(z)}(x)$ exists and equals 0. (Formally, it is only the latter assumption that matters in the proof of Theorem 4.2 in the next sub-section.)

Likewise, we suppose that $\forall\ x \in \mathbb{X}$ there is a closed linear operator $\mathtt{A}^{(x)}$ acting on functions $f: \mathcal{Z} \to \mathbb{R}$ forming a domain $\mathrm{D}(\mathtt{A}^{(x)})$ dense in $\mathrm{C_b}(\mathcal{Z})$, the space of bounded continuous functions on $\mathcal{Z}$. In addition, we assume that each $\mathtt{A}^{(x)}$ is a generator of an Feller MP on $\mathcal{Z}$. We also suppose that $\forall\ f$ from a core of $\mathtt{A}^{(x)}$, the integral $\int_{\mathcal{Z}}(\mathtt{A}^{(x)}f)(z)\mathrm{d}\upsilon(z)$ exists and is equal to 0. (Note that measure υ serves all operators $\mathtt{A}^{(x)}$.)

4.C. A combined generator. We are now going to introduce generator $\mathbf{R}$ and analyse the related WIE. Fix a time-scale function $z \in \mathcal{Z} \mapsto \sigma(z)$ with $\sigma(z) > 0$ for υ-a.a. $z \in \mathcal{Z}$. Consider the following measure $\boldsymbol{\kappa}$ on $\mathcal{Z} \times \mathbb{X}$:

$$\boldsymbol{\kappa}(B) = \int \frac{\mathbf{1}((z,x) \in B)}{\sigma(z)} m(z,x)\mathrm{d}\gamma(x)\mathrm{d}\upsilon(z),\ \forall\ \text{Borel}\ \ B \subseteq \mathcal{Z} \times \mathbb{X}. \tag{4.4}$$

Then introduce a linear map $\mathbf{R}$ acting on functions $(z,x) \in (\mathcal{Z} \times \mathbb{X}) \mapsto \phi(z,x) \in \mathbb{R}$, $\phi \in \mathrm{D}(\mathbf{R})$, where

$$\mathbf{R}\phi(z,x) = \mathbf{A}^{(x)}\phi(z,x) + \mathbf{L}^{(z)}\phi(z,x). \tag{4.5}$$

Here, it is understood that $\mathbf{A}^{(x)}\phi(z,x)$ results in the action of $\mathtt{A}^{(x)}$ in SE-variable z succeeded by multiplication by $1/m(z,x)$, and $\mathbf{L}^{(z)}\phi(z,x)$ results in $\mathtt{L}^{(z)}$ acting in variable $x \in \mathbb{X}$. Formally, for $\phi = f \otimes g$, with $\phi(z,x) = f(z)g(x)$:

$$[\mathbf{A}^{(x)}(f \otimes g)](z,x) = g(x)\frac{\sigma(z)(\mathtt{A}^{(x)}f)(z)}{m(z,x)},\ \ \big[\mathbf{L}^{(z)}(f \otimes g)\big](z,x) = \alpha(z)f(z)(\mathtt{L}^{(z)}g)(x) \tag{4.6}$$

where $z \in \mathcal{Z} \mapsto \alpha(z) > 0$ is another given time-scale function. Referring to a Banach space $\mathrm{C_b}(\mathcal{Z}) \otimes \mathrm{C_b}(\mathbb{X})$, with a chosen cross-norm, the action of $\mathbf{R}$ is then extended by linearity and continuity to general functions ϕ.

In this paper we do not intend to go into details of formal constructions of a closed operator based on map $\mathbf{R}$ defined in (4.5). Nor shall we try to analyse the conditions of the Hille–Yosida theorem for such an operator. (Still, we will refer to this operator as $\mathbf{R}$ and suppose that it exists.) It will be assumeed that the domain $\mathrm{D}(\mathbf{R})$ contains functions $\phi \in \mathrm{C_b}(\mathcal{Z}) \otimes \mathrm{C_b}(\mathbb{X})$ such that (i) $\forall\ z \in \mathcal{Z}$, function $g_{\phi,z}:\ x \mapsto \phi(z,x)$ belongs to $\mathrm{D}(\mathtt{L}^{(z)})$ (ii) $\forall\ x \in \mathbb{X}$, function $f_{\phi,x}:\ z \mapsto \phi(z,x)$ belongs to $\mathrm{D}(\mathtt{A}^{(x)})$. (We will say that functions $f_{\phi,x}$ and $g_{\phi,z}$ give section maps generated by function ϕ.) Pictorially, we would like to treat $\mathbf{R}$ as a generator of an MP $(Z(t), X(t))$ on $\mathcal{Z} \times \mathbb{X}$ which is a superposition of two tendencies: one is to keep an SE value z intact and evolve in component $\mathbb{X}$ in accordance with the MP generated by $\alpha(z)\mathtt{L}^{(z)}$, the other to keep state x of the base MP but change the SE value z by following the MP on $\mathcal{Z}$ generated by $\sigma(z)\mathtt{A}^{(x)}$. (As examples in this paper show, under additional assumptions such a process can be constructed.)

For functions ϕ satisfying (i), (ii), formula (4.5) becomes

$$\mathbf{R}\phi(z,x) = [\mathbf{A}^{(x)}f_{\phi,x}](z) + [\mathbf{L}^{(z)}g_{\phi,z}](x).$$

Theorem 4.2. *The following properties* (I)–(II) *are satisfied.* (I) *Measure* $\boldsymbol{\kappa}$ *and operator* $\mathbf{R}$ *from* (4.4)–(4.6) *obey the WIE in the sense that the integral* $\int_{\mathcal{Z}\times\mathbb{X}}(\mathbf{R}\phi)\mathrm{d}\boldsymbol{\kappa}$ *exists and equals* 0 $\forall$ *function* ϕ *for which*

$$\int_{\mathbb{X}}\left\{\int_{\mathcal{Z}}\Big|\left[\mathtt{A}^{(x)}f_{\phi,x}\right](z)\Big|\mathrm{d}\upsilon(z)\right\}\mathrm{d}\gamma(x)<\infty\ \ \textit{and}\ \ \int_{\mathcal{Z}}\left[\mathtt{A}^{(x)}f_{\phi,x}\right](z)\mathrm{d}\upsilon(z)=0,\ \textit{for}\ \gamma\text{-a.a. } x\in\mathbb{X},$$

and

$$\int_{\mathcal{Z}}\frac{\alpha(z)}{\sigma(z)}\left\{\int_{\mathbb{X}}\Big|\left[\mathtt{L}^{(z)}g_{\phi,z}\right](x)\Big|\mathrm{d}\nu^{(z)}(x)\right\}\mathrm{d}\upsilon(z)<\infty\ \ \textit{and}\ \ \int_{\mathbb{X}}\left[\mathtt{L}^{(z)}g_{\phi,z}\right](x)\mathrm{d}\nu^{(z)}(x)=0,\ \textit{for}\ \upsilon\text{-a.a. } z\in\mathcal{Z}.$$

(II) *If* $\Xi=\boldsymbol{\kappa}(\mathcal{Z}\times\mathbb{X})<\infty$ *then* $\boldsymbol{\pi}(B)=\boldsymbol{\kappa}(B)/\Xi$ *yields a probability distribution on* $\mathcal{Z}\times\mathbb{X}$, *with* $\int_{\mathcal{Z}\times\mathbb{X}}(\mathbf{R}\phi)\mathrm{d}\boldsymbol{\pi}=0\ \forall\ \phi$ *as in the above definition.*

Proof. As before, we focus on assertion (I). Here, for a function ϕ, satisfying the conditions of the theorem,

$$\int_{\mathcal{Z}\times\mathbb{X}}(\mathbf{R}\phi)\mathrm{d}\boldsymbol{\kappa}=\int_{\mathbb{X}}\left\{\int_{\mathcal{Z}}\left[\mathtt{A}^{(x)}f_{\phi,x}\right](z)\mathrm{d}\upsilon(z)\right\}\mathrm{d}\gamma(x)+\int_{\mathcal{Z}}\frac{\alpha(z)}{\sigma(z)}\left\{\int_{\mathbb{X}}\left[\mathtt{L}^{(z)}g_{\phi,z}\right](x)\mathrm{d}\nu^{(z)}(x)\right\}\mathrm{d}\upsilon(z).$$

Both summands in the RHS vanish owing to the assumptions. □

Remarks. 4.3. As in previous results, the time-scaling factor α does not enter expression for the IM in (4.4).

4.4. Operators $\mathtt{A}^{(x)}$ enter Eqn (4.4) via a *common* IM υ. This fact leads to a time-homogeneous combined MP.

4.5. The condition that $\nu^{(z)}<<\gamma$ and $m(z,x)>0$ guarantees that the change of environment results in a 'non-disruptive' continuation of the combined MP (should it exist).

4.6. Like earlier parts of the exposition, additional assumptions are needed if we wish the combined MP to exist (some of these assumptions are hard to verify). Viz., conditions guaranteeing that a sum of two generators is a generator of a continuous contraction semi-group are listed in [3], Theorem 7.1, P. 37 and Corollary 7.2, P. 38. Sufficient conditions under which an operator generates a continuous positive contraction semigroup are given in [3], Theorem 2.2, P. 165. The fact that a Feller semi-group (i.e., continuous positive contraction semi-group with a conservative generator) induces an MP with Skorokhod-type sample paths is established in [3], Theorem 2.7, P.169 and Corollary 2.8, P. 170. The equivalence of the WIE and the IM property is proved, under additional assumptions, in [3], Proposition 9.2, P. 239. See also [3], Theorem 9.17, P. 248. With regard of a sum of two generators, other relevant results are contained in [3], Section 10 of Chapter 4, PP. 253–261.

4.7. We would like to point at an iterative property of our construction: once constructed, a combined MP may be used as a basic/environment MP to produce a combined process of a 'higher' level.

4.8. The Markov property of basic processes can be replaced by weaker assumptions covering a broader classes of random processes with an infinite memory. This can be a topic for future research.

5 Processes with diffusive components

Diffusion MPs play an important role in the modern theory as they are described in comprehensible terms and provide a broad spectrum of interesting properties. A particular feature of diffusion processes is that they are generated by second-order differential operators, and existence and properties of an MP are expressed here through properties of coefficients and boundary conditions. In this section we comment on examples of the above construction where basic MPs or an environment MP or both are represented by diffusions.

5.A. Models with a jump basic process. In the current sub-section we discuss models where the basic process is a jump MP whereas the SE (state of environment) process is a diffusion. More precisely, we deal with simple, but not entirely trivial, examples of an isolated $M/M/1/\infty$ queue, with $\mathbb{X} = \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. We employ the notation traditionally used in the literature in this area. (So, λ stands for the Poissonian arrival rate and μ for the service rate; condition $\lambda < \mu$ is necessary and sufficient for the queue to be stable.)

5.A.1. First, take $\mathcal{Z} = (\epsilon, 1)$ where $0 < \epsilon < 1$. Here the SE-point $z = \lambda$ follows a diffusion process with coefficient $\beta(n) > 0$ and with reflections at the borders ϵ and 1. For the basic MP we have jump rates corresponding with $\mu = 1$:

$$Q^{(\lambda)}(n, n+1) = \lambda, \;\; Q^{(\lambda)}(n, n-1) = \mathbf{1}(n \geq 1), \;\; n \in \mathbb{Z}_+.$$

In other words, we fix the service rate $\mu = 1$ and vary the arrival rate λ between ϵ and 1 as prescribed by the above diffusion. The IM υ on $(\epsilon, 1)$ is Lebesgue: $\mathrm{d}\upsilon(\lambda) = \mathrm{d}\lambda$, while on $\mathbb{Z}_+$ the IM $\nu^{(\lambda)}$ is geometric, with $\nu^{(\lambda)}(n) = \lambda^n$. (For the sake of convenience, we omit the normalizing factor.) The combined state space is $(0, 1) \times \mathbb{Z}_+$, and the combined generator $\mathbf{R}$ acts by

$$\begin{aligned}\mathbf{R}\phi(\lambda, n) \;\; &= \frac{\sigma(\lambda)\beta(n)^2}{2\lambda^n} \frac{\partial^2}{\partial \lambda^2} \phi(\lambda, n) \\ &+ \alpha(\lambda) \Big\{ \lambda \big[\phi(\lambda, n+1) - \phi(\lambda, n) \big] + \big[\phi(\lambda, n-1) - \phi(\lambda, n) \big] \mathbf{1}(n \geq 1) \Big\}.\end{aligned} \tag{5.1}$$

The domain $\mathrm{D}(\mathbf{R})$ consists of functions $(\lambda, n) \mapsto \phi(\lambda, n)$ that are C^2 in variable $\lambda \in (\epsilon, 1)$ and satisfy the Neumann boundary condition $\dfrac{\partial}{\partial \lambda}\phi(\epsilon+, n) = \dfrac{\partial}{\partial \lambda}\phi(1-, n) = 0$. Assuming that (i) $\sigma(\lambda)$ is continuous and strictly positive on $[\epsilon, 1]$ and (ii) $\sum\limits_{n \in \mathbb{Z}_+} \beta(n)^2/\epsilon^n < \infty$, it is possible to check that operator $\mathbf{R}$ from Eqn (5.1) generates a Feller semi-group MP on $\mathrm{C_b}([\epsilon, 1] \times \mathbb{Z}_+)$; cf. [8], Section 2.1, Chapter 2, or [12], Sections 3 and 4, Chapter 15. The IMs for the corresponding MP on $(Z(t), X(t))$ with generator (5.1) are analyzed in Theorem 5.1 below.

Theorem 5.1. *Suppose that conditions* (i) *and* (ii) *are fulfilled.* (I) *An IM on* $(0,1)\times\mathbb{Z}_+$ *has the form* $\boldsymbol{\kappa}(A,n)=\int_A \dfrac{\lambda^n}{\sigma(\lambda)}\mathrm{d}\lambda$, $A\subseteq(\epsilon,1)$. (II) *If* $\Xi:=\sum\limits_{n\geq 0}\int_\epsilon^1\dfrac{\lambda^n}{\sigma(\lambda)}\mathrm{d}\lambda<\infty$ *(the SCC), the normalized measure* $\boldsymbol{\pi}(A,n)=\boldsymbol{\kappa}(A,n)/\Xi$ *yields an EPD, and process* $(Z(t),X(t))$ *is positive recurrent in the sense that* $\forall\ n\in\mathbb{Z}_+$ *and interval* $A\subset(\epsilon,1)$ *the mean return time to the set* $A\times\{n\}$ *is finite.*

Proof. As above, (II) is a technical corollary, and we focus on assertion (I). Here our task is to check that measure $\boldsymbol{\kappa}$ is annihilated by the conjugate $\mathbf{R}^*$. The corresponding calculation is straightforward: the shortest proof is to pass to the density $\dfrac{\lambda^n}{\sigma(\lambda)}=:\dfrac{\boldsymbol{\kappa}(\mathrm{d}\lambda,n)}{\mathrm{d}\lambda}$ and check that $\sum\limits_{n\geq 0}\int_\epsilon^1\dfrac{\lambda^n}{\sigma(\lambda)}(\mathbf{R}f)(\lambda,n)\mathrm{d}\lambda=0\ \forall\ f\in\mathrm{D}(\mathbf{R})$ for which the sum $\sum\limits_{n\geq 0}\int_\epsilon^1\dfrac{\lambda^n}{\sigma(\lambda)}\big|(\mathbf{R}f)(\lambda,n)\big|\mathrm{d}\lambda<\infty$. $\square$

E.g., with $\sigma(\lambda)\equiv 1$ (no time change in the environment MP), we have $\Xi=\sum\limits_{n\geq 1}\dfrac{1-\epsilon^n}{n}=\infty$, and assertion (II) does not provide an EPD. However, for $\sigma(\lambda)=\dfrac{1}{1-\lambda}$ (an acceleration/chaotization for $\lambda\sim 1$),

$$\Xi=\sum_{n\geq 0}\int_\epsilon^1\lambda^n(1-\lambda)\mathrm{d}\lambda=\sum_{n\geq 1}\left(\frac{1-\epsilon^n}{n}-\frac{1-\epsilon^{n+1}}{n+1}\right)<\infty,$$

and the construction in assertion (II) guarantees existence of an EPD.

5.A.2. Our next example is a dual of the previous one: now we fix a value for the arrival rate $\lambda=1$ and let SE $z=\mu$ follow a Brownian motion on $\mathcal{Z}=(1,\infty)$, with a drift b and a reflection at the leftmost point 1. The IM $\mathrm{d}\upsilon(\mu)$ on $(1,\infty)$ is $e^{2(\mu-1)b}\mathrm{d}\mu$ (which is just Lebesgue when $b=0$). The combined generator $\mathbf{R}$ acts on C^2-functions $(\mu,n)\in(1,\infty)\times\mathbb{Z}_+\mapsto\phi(\mu,n)$ with $\dfrac{\partial}{\partial\mu}\phi(1+,n)=0$:

$$\begin{aligned}\mathbf{R}\phi(\mu,n)\ &=\mu^n\sigma(\mu)\left[\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\phi(\mu,n)+b\frac{\partial}{\partial\mu}\phi(\mu,n)\right]\\ &\quad+\alpha(\mu)\Big\{\big[\phi(\mu,n+1)-\phi(\mu,n)\big]+\mu\big[\phi(\mu,n-1)-\phi(\mu,n)\big]\mathbf{1}(n\geq 1)\Big\}.\end{aligned}\tag{5.2}$$

In this setting, we leave open the question of existence of the combined MP, focusing instead on the weak invariance equation (WIE). We say that a function $(\mu,n)\in(1,\infty)\times\mathbb{Z}_+\mapsto\eta(\mu,n)$ (with real values $\eta(\mu,n)\in\mathbb{R}$) satisfies the WIE with generator $\mathbf{R}$ from (5.2) if, $\forall\ (\mu,n)\in(1,\infty)\times\mathbb{Z}_+$, the following properties hold: (i)

$$\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\Big[\mu^n\sigma(\mu)\eta(\mu,n)\Big]-b\frac{\partial}{\partial\mu}\Big[\mu^n\sigma(\mu)\eta(\mu,n)\Big]=0,$$

and (ii)

$$\eta(\mu,n)\big[1+\mu\mathbf{1}(n\geq 1)\big]=\eta(\mu,n-1)\mathbf{1}(n\geq 1)+\mu\eta(\mu,n+1).$$

Theorem 5.2. (I) *The function* $(\mu, n) \in (1, \infty) \times \mathbb{Z}_+ \mapsto \kappa(\mu, n)$ *of the form* $\kappa(\mu, n) = \dfrac{e^{2(\mu-1)b}}{\mu^n \sigma(\mu)}$ *satisfies the WIE with generator* $\mathbf{R}$ *from* (5.2). (II) *Under the SCC* $\Xi := \sum_{n\geq 0} \int_1^\infty \dfrac{e^{2(\mu-1)b}}{\mu^n\sigma(\mu)}\mathrm{d}\mu < \infty$, κ *yields a probability density function.*

The proof is done by a direct substitution and is omitted.

Again, with $\sigma(\mu) \equiv 1$, the partition function $\Xi = \int_1^\infty \dfrac{\mu e^{2(\mu-1)b}}{\mu - 1}\mathrm{d}\mu = \infty$, regardless of the value b. To obtain a PDF in assertion (II), we need to introduce a time-scale $\sigma(\mu)$ growing at $\mu \sim 1$ and – when $b \geq 0$ (i.e., no the drift towards 1) – at $\mu \sim \infty$.

5.A.3. Now consider the case where the SE is a pair (λ, μ) varies according to a joint diffusion in a space $\mathcal{Z}$ identified as a $\pi/4$-angle $\mathcal{A} = \{(\lambda, \mu) : \ \mu > \lambda > 0\}$, with drift $\begin{pmatrix} \theta \\ \theta \end{pmatrix}$, reflected at the sides $\lambda = 0$ and $\lambda = \mu$, along the inward normal directions. It means that generator $\mathtt{A}^{(x)} = \mathtt{A}$ acts by

$$\mathtt{A}f(\lambda, \mu) = \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}f(\lambda, \mu) + \frac{1}{2}\frac{\partial^2}{\partial\mu^2}f(\lambda, \mu) + \theta\frac{\partial}{\partial\lambda}f(\lambda, \mu) + \theta\frac{\partial}{\partial\mu}f(\lambda, \mu). \tag{5.3}$$

The domain of $\mathtt{A}$ consists of C^2-functions $(\lambda, \mu) \in \mathcal{A} \mapsto f(\lambda, \mu)$ satisfying the boundary conditions $\dfrac{\partial}{\partial\lambda}f(0+, \mu) = 0$ and $\left(\dfrac{\partial}{\partial\lambda} - \dfrac{\partial}{\partial\mu}\right) f(\lambda, \mu)\Big|_{\lambda=\mu} = 0$. Here the IM υ has the Lebesgue density

$$\frac{\mathrm{d}\upsilon(\lambda, \mu)}{\mathrm{d}\lambda\mathrm{d}\mu} = \exp\left[2\theta(\lambda + \mu)\right], \quad (\lambda, \mu) \in \mathcal{A}. \tag{5.4}$$

(A finite IM arises iff $\theta < 0$.)

Accordingly, the generator $\mathbf{R}$ has the form

$$\begin{aligned} &\mathbf{R}\phi(\lambda, \mu, n) \\ &\quad = \frac{\mu^n\sigma(\lambda, \mu)}{\lambda^n}\left[\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\phi(\lambda, \mu, n) + \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\phi(\lambda, \mu, n) + \theta\frac{\partial}{\partial\lambda}\phi(\lambda, \mu, n) + \theta\frac{\partial}{\partial\mu}\phi(\lambda, \mu, n)\right] \\ &\qquad +\alpha(\lambda, \mu)\Big\{\big[\phi(\lambda, \mu, n+1) - \phi(\lambda, \mu, n)\big] + \mu\big[\phi(\lambda, \mu, n-1) - \phi(\lambda, \mu, n)\big]\mathbf{1}(n \geq 1)\Big\} \end{aligned} \tag{5.5}$$

and acts on functions $(\lambda, \mu, n) \in \mathcal{A} \times \mathbb{Z}_+ \mapsto \phi(\lambda, \mu, n)$ which are C^2 and satisfy the above boundary conditions in variables λ, μ. Repeating the previous defintion *mutatis mutandis* yields

Theorem 5.3. (I) *Consider the function* $(\lambda, \mu, n) \in \mathcal{A} \times \mathbb{Z}_+ \mapsto \kappa(\lambda, \mu, n)$ *of the form*

$$\kappa(\lambda\mu, n) = \frac{\lambda^n \exp\left[2\theta(\lambda + \mu)\right]}{\mu^n\sigma(\lambda, \mu)}.$$

Then κ *satisfies the WIE with the generator* $\mathbf{R}$ *from* (5.5).

(II) *Assuming the SCC* $\Xi := \sum_{n\geq 0}\int_{\mathcal{Z}} \dfrac{\lambda^n \exp\left[2\theta(\lambda+\mu)\right]}{\mu^n\sigma(\lambda,\mu)}\mathrm{d}\lambda\mathrm{d}\mu < \infty$, *the normalized function* $\kappa(\lambda,\mu,,n)\big/\Xi$ *yields a PDF on* $\mathcal{A}\times\mathbb{Z}_+$.

Remarks. 5.1. The indicated form of IM υ on $\mathcal{Z}=\mathcal{A}$ in Eqn (5.4) can be obtained by observing that the SE diffusion process is a projection, upon $\mathcal{A}$, of a 'covering' MP living in a quadrant $\mathcal{B}=\{(\lambda,\mu):\lambda,\mu>0\}$. (The projection is $(\lambda,\mu)\in\mathcal{B}\mapsto(\lambda\wedge\mu,\lambda\vee\mu)$ where $\lambda\wedge\mu=\min[\lambda,\mu]$, $\lambda\vee\mu=\max[\lambda,\mu]$.) The covering MP is a diffusion with the same drift vector (parallel to the bissectrice) and the normal reflections from the sides $\lambda=0$ and $\mu=0$. It is easy to see that the covering diffusion is a product of two 1D diffusions, one in λ and the other in μ, each on $\mathbb{R}_+=(0,\infty)$, with drift θ, reflection at the origin. A more general class of boundary conditions can also be considered, by following results from [5]–[6], [18].

5.2. The methodology developed thus far in this section allows us to proceed with the case of a Jackson network. Here it is convenient to pass to vectors $\underline{\rho}$ and $\underline{\mu}$ and consider (individual or joint) diffusions in the corresponding wedge-like domains. (Matrix $\mathtt{P}$ can also be varied in its own simplex-type domain.) The resulting picture essentially looks like the one above; detailed questions are left for/can be a subject of future research.

5.B. Models with a basic diffusion process. Models with basic diffusion MPs and a jump change of the environment are quite popular in some chapters of the theory of controlled diffusion processes (switching diffusions); see [1], [16], [19]. Here we discuss a straightforward example where $\mathbb{X}=\mathbb{R}$. The basic MP lives in a line and has diffusion coefficient 1 and a drift $z\in\mathbb{R}$; the latter is considered as an SE. Consequently, $\mathrm{d}v(x)=\mathrm{d}x$, and the IM $\nu^{(z)}(\mathrm{d}x)=e^{2zx}\mathrm{d}x$ with $m(z,x)=e^{2zx}$, $x\in\mathbb{R}$. Next, we take $\mathcal{Z}=\mathbb{R}$ and consider a jump environment MP with a jump measure $T^{(x)}(z,\mathrm{d}z')$. Let υ be an IM with $\int_{\mathbb{R}}\upsilon(\mathrm{d}z)T^{(x)}(z,\mathrm{d}z')=\upsilon(\mathrm{d}z')$. The individual components become

$$\mathtt{L}^{(z)}g(x)=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)+z\frac{\mathrm{d}}{\mathrm{d}x}g(x),\quad m(z,x)=e^{2zx},\quad z,x\in\mathbb{R},\tag{5.6}$$

$$\mathtt{A}^{(x)}f(z)=\int_{\mathbb{R}}T^{(x)}(z,\mathrm{d}z')[\phi(x,z')-\phi(x,z)],\quad z,x\in\mathbb{R}.\tag{5.7}$$

Then set

$$\begin{array}{r}\mathrm{d}\boldsymbol{\kappa}(z,x)=e^{2vz}\upsilon(\mathrm{d}z)\mathrm{d}x,\\ \mathbf{R}\phi(z,x)=\alpha(z)\left[\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\phi(z,x)+z\dfrac{\partial}{\partial x}\phi(z,x)\right]\\ +\sigma(z)e^{-2zx}\int_{\mathbb{R}}T^{(x)}(z,\mathrm{d}z')[\phi(x,z')-\phi(x,z)],\end{array}\qquad x,z\in\mathbb{R}.\tag{5.8}$$

$\Big($A simple example is where $z,z'=\pm1$ with jump intensities $T^{(x)}(1,-1)=T^{(x)}(-1,1)=q(x)$; here the IM $\upsilon=b(\delta_1+\delta_{-1})$, the sum of Dirac deltas supported at $z=\pm1$, with equal coefficients. Another

example is where $T^{(x)}(z, \mathrm{d}z') = T^{(x)}(z+u, \mathrm{d}(z'+u))\ \ \forall\ u, z, z' \in \mathbb{R}$ (shift-invariant jump measures). Here $\upsilon(\mathrm{d}z) = \mathrm{d}z$ is Lebesgue.$\Big)$

Theorem 5.4. (I) *Operator* $\mathbf{R}$ *and measure* $\boldsymbol{\kappa}$ *satisfy the WIE on* $\mathbb{R} \times \mathbb{R}$. (II) *Under the SCC* $\Xi := \int_{\mathbb{R}\times\mathbb{R}} \dfrac{e^{2zx}}{\sigma(z)} \mathrm{d}x\upsilon(\mathrm{d}z) < \infty$, *we obtain a unique PD satisfying the WIE.*

Remarks. 5.3. As above, one can refer to Theorem 3.1, P. 377 in [3], and provide sufficient conditions upon $T^{(x)}(z, \mathrm{d}z')$ under which the combined MP will exist, with a Feller transition semigroup on $\mathbb{R} \times \mathbb{R}$.

5.4. The SCC $\Xi < \infty$ requires a rapid growth of $\sigma(z)$ at $z \sim \pm\infty$.

5.C. Models of a two-component diffusion. The next class of models is where both basic and SE MPs are diffusions. The simplest model is where the basic MP $\widetilde{X}^{(z)}$ is a d-dimensional time-scaled Wiener process (WP), with $\mathbb{X} = \mathbb{R}^d$, the SE space $\mathcal{Z} = \mathbb{R}$, and the SE process $\widetilde{Z}(t)$ is a standard WP on a line. Here

$$\begin{aligned} \mathtt{L}^{(z)} g(x) &= \frac{z^2}{2} \sum_{1 \le i \le d} \frac{\partial^2}{\partial \mathrm{x}_i^2} g(x), \\ \mathtt{A} f(z) &= b f'(z) + \frac{1}{2} f''(z), \end{aligned} \qquad x = (\mathrm{x}_1, \ldots, \mathrm{x}_d) \in \mathbb{R}^d,\ z \in \mathbb{R}. \tag{5.9}$$

Moreover, the reference measure $\mathrm{d}\gamma(x)$ is d-dimensional Lebesgue, $\mathrm{d}x$, and the IM $\mathrm{d}\upsilon(z)$ is absolutely contiunous wrt $\mathrm{d}z$, one-dimensional Lebesgue:

$$\begin{gathered} m(z, x) = 1 \ \text{ that is, } \mathrm{d}\nu^{(z)}(x) = \mathrm{d}x, \\ \mathrm{d}\upsilon(z) = e^{2bz} \mathrm{d}z. \end{gathered} \tag{5.10}$$

Both $\nu^{(z)}$ and υ are genuine IMs for their respective MPs $\widetilde{X}^{(z)}$ and $\widetilde{Z}(t)$.

Next, fix coefficient functions $z \in \mathbb{R}_+ \mapsto \alpha(z) > 0$ and $z \in \mathbb{R}_+ \mapsto \sigma(z) > 0$ such that

$$\alpha \text{ and } \sigma \text{ are Hoelder functions, and } c < z^2 \alpha(z), \sigma(z) < C(1 + z^2) \tag{5.11}$$

where $c, C \in (0, \infty)$ are constants. The combined generator $\mathbf{R}$ has the form

$$\mathbf{R}\phi(z, x) = \alpha(z) \frac{z^2}{2} \sum_{1 \le i \le d} \frac{\partial^2}{\partial \mathrm{x}_i^2} \phi(z, x) + \sigma(z) \left[b \frac{\partial}{\partial z} \phi(z, x) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \phi(z, x) \right],\ x \in \mathbb{R}^d,\ z \in \mathbb{R}, \tag{5.12}$$

and acts on C^2-functions $(z, x) \mapsto \phi(z, x)$. More precisely, $\mathbf{R}$ is a closed operator whose domain $\mathrm{D}(\mathbf{R})$ is the closure, in $\mathrm{C_b}(\mathbb{R}_+ \times \mathbb{R}^d)$, of the set of functions ϕ such that $\mathbf{R}\phi \in \mathrm{C_b}(\overline{\mathbb{R}}_+ \times \mathbb{R}^d)$. Owing to a general result, such as Theorem 1.5 on P. 369 in [3], Section 8.1; see also Ref. [11], Sections 5.2–5.4,

$\mathbf{R}$ gives rise to a unique Feller semi-group of operators described by transition densities (relative to $\mathrm{d}z \times \mathrm{d}x$). Consequently, there exists a unique two-component diffusion process $(Z(t), X(t))$ in $\mathbb{R}_+ \times \mathbb{R}^d$ generated by $\mathbf{R}$. Then, in accordance with Proposition 9.2 in [3], measure $\boldsymbol{\kappa}$ with

$$\boldsymbol{\kappa}(\mathrm{d}z \times \mathrm{d}x) = \frac{e^{2bz}}{\sigma(z)}\mathrm{d}z\mathrm{d}x \tag{5.13}$$

yields a genuine IM for $(Z(t), X(t))$. As the basic IM is Lebesgue, $\Xi := \int_{\mathbb{R}_+\times\mathbb{R}} \frac{e^{2bz}}{\sigma(z)}\mathrm{d}z\mathrm{d}x = \infty$, and there is no EPD.

The two-component diffusion can be described as a solution to a system of stochastic integro-differential equations (SIDEs)

$$\begin{array}{c} \mathrm{d}X(t) = \sqrt{\alpha(Z(t))}Z(t)\mathrm{d}W_1(t),\ X(0) = x^0 \in \mathbb{R}^d, \\ \mathrm{d}Z(t) = b\sigma(Z(t))dt + \sqrt{\sigma(Z(t))}\mathrm{d}W_2(t) + \mathrm{d}L_Z(t),\ Z(0) = z^0 > 0, \\ L_Z(t) = \int_0^t \mathbf{1}(Z(s) = 0)\mathrm{d}L_Z(s). \end{array} \tag{5.14}$$

Here $W_1(t)$ is the WP in $\mathbb{R}^d$ and $W_2(t)$ the WP in $\mathbb{R}$, and $W_1(t)$ and $W_2(t)$ are independent. Process $L_Z(t)$ is a local time spend by $Z(s)$ at 0 by time t. ($L(t)$ increases on a set of time points of measure 0.)

Remark 5.4. SDEs (5.14) involve the solution to the so-called Skorohod problem. Here we consider a WP $W(t)$ on $\mathbb{R}$ and construct a BM $Z(t)$ on $(0, \infty)$ with reflection at 0 by taking a pair of processes $(Z(t), L(t))$ such that (i) $Z(t) = z^0 + W(t) + L(t)$, $t \geq 0$, $z^0 > 0$, (ii) $Z(t) \geq 0$, $t \geq 0$, and (iii) process $L(t)$ has $L(0) = 0$ (starts at 0), is continuous, monotone nondecreasing and obeys $\mathrm{supp}\,(\mathrm{d}L) \subset \{t \geq 0 : Z(t) = 0\}$ (that is, $L(t)$ increases only at times when $Z(t)$ is equal to zero). Moreover, (iv) $L(t) = [-z^0 - \inf_{s\leq t} W(s)]^+$. In addition, (v) the distributions of processes $Z(t)$ and $|W(t)|$ coincide.

In a similar way, for $\mathcal{Z} = (0, 1)$ the solution of the Skorohod problem is a triple of processes $(Z(t), L(t), U(t))$ such that (i) $Z(t) = z^0 + W(t) + L(t) - U(t)$, $t \geq 0$, $0 < z^0 < 1$, (ii) $0 \leq Z(t) \leq 1$, $t \geq 0$, (iii) process $L(t)$ has $L(0) = 0$, is continuous, monotone nondecreasing and obeys $\mathrm{supp}\,(\mathrm{d}L) \subset \{t \geq 0 : Z(t) = 0\}$, and (iv) a similar property holds for $U(t)$ (repacing endpoint 0 with 1).

An analogous construction works for $\mathcal{Z} = (z^1, z^2)$.

6 Two-component diffusions based on an Ornstein–Uhlenbeck process

In this section we consider several examples of combined two-dimensional diffusion where the basic MP is an Ornstein–Uhlenbeck process (OUP). Such examples may be of interest in financial calculus.

In all models that are discussed here, (a) we take $\alpha(z)=\sigma(z)\equiv 1$, (b) the SE generator $\mathtt{A}^{(x)}=\mathtt{A}$ is independent of x, (c) space $\mathbb{X}$ is a line $\mathbb{R}$ or a bounded interval $\mathbb{I}=(x^1,x^2)\subset\mathbb{R}$, with Lebesgue measure $\mathrm{d}\gamma(x)=\mathrm{d}x$, and (d) space $\mathcal{Z}$ coincides with a half-line $\mathbb{R}_+=(0,\infty)$ or the whole line $\mathbb{R}$ or a bounded interval $\mathbb{J}=(z^1,z^2)\subset\mathbb{R}$, and (e) υ is an absolutely continuous measure υ: $\mathrm{d}\upsilon(z)=w(z)\mathrm{d}z$. (We will loosely refer to $\mathcal{Z}$ as an interval.) The basic MP $\widetilde{X}^{(z)}(t)$ on $\mathbb{X}$ has generator $\mathtt{L}^{(z)}$ where, for a C^2-function $x\in\mathbb{X}\mapsto g(x)$,

$$\mathtt{L}^{(z)}g(x)=-xg'(x)+\frac{z^2}{2}g''(x),\ \text{ with }\ m(z,x)=e^{-x^2/z^2},\ \ x\in\mathbb{X},\ z\in\mathcal{Z}. \tag{6.1}$$

More precisely, let $\mathrm{C_b}=\mathrm{C_b}(\mathbb{X})$ denote the space of bounded continuous functions $x\in\mathbb{X}\mapsto g(x)$ with the sup-norm. The (closed) operator $\mathtt{L}^{(z)}$ acts on its domain $\mathrm{D}(\mathtt{L}^{(z)})\subset\mathrm{C_b}(\mathbb{X})$ which is the closure of the set of C^2 functions g such $\mathtt{L}^{(z)}g\in\mathrm{C_b}(\mathbb{X})$ and g satisfies a chosen boundary condition at the endpoints in case $\mathbb{X}=\mathbb{I}$. The coefficient z^2 is interpreted as a stochastic volatility. In all models under consideration, there exists a Feller semi-group generated by $\mathtt{L}^{(z)}$ which is determined by transition probability densities. Viz., for $\mathbb{X}=\mathbb{R}$, the densities are

$$p_t(x,x')=\frac{1}{\sqrt{\pi}|z|}\exp\left[-\frac{(x'-xe^{-t})^2}{z^2(1-e^{-2t})}\right],\ \ x,x'\in\mathbb{R}. \tag{6.2}$$

In this case, after normalization, the IM density $m(z,x)=e^{-x^2/z^2}$, $z\neq 0$, gives rise to a Gaussian probability distribution $\mathrm{N}(0,z^2/2)$.

Examples of the SE diffusion in this section will be a Brownian motion (BM) on $\mathbb{R}_+$ with a reflection at 0 (sub-section 6.B), an OUP on $\mathbb{R}$ (sub-section 6.C) and an affine Cox–Ingersoll–Ross diffusion on $\mathbb{R}_+$ (sub-section 6.D), as well as versions of these models on bounded intervals $\mathbb{J}$.

The existence/uniqueness of combined Feller MPs on bounded rectangles $\mathbb{I}\times\mathbb{J}$ will follow from regularity (Hoelderness and non-degeneracy) of coefficients/boundary conditions. See, Ref. [3], Section 8.1, Theorem 1.5 on P. 369, or Ref. [11], Sections 5.2–5.4. Consequently, for such examples covered by these general results we will be able to use the term IM unreservedly. Otherwise we only work with generator $\mathbf{R}$ (see Eqn (6.4)) and will have to employ the WIEs (weak-invariance equations). Cf. Eqn (6.11) below.

6.A. Let us first re-formulate our general construction in the situation where basic and environment MPs are diffusions. As was said, we consider the case where the dimension of each component equals 1; however, the general scheme can be also developed in a multi-dimensional setting. The attention is on a two-component diffusion MP $(Z(t),X(t))$ in a Cartesian product $\mathcal{Z}\times\mathbb{X}$ where $\mathbb{X}$ and $\mathcal{Z}$ are of the types described above. Here, the Lebesgue measures $\mathrm{d}z$ and $\mathrm{d}x$ will play special roles.

The coefficient functions

$$x\in\mathbb{R}\mapsto a(x)\in\mathbb{R},\ z\in\mathcal{Z}\mapsto c(z)\in\mathbb{R},\ z\in\mathcal{Z}\mapsto C(z)>0,\ (z,x)\in\mathcal{Z}\times\mathbb{X}\mapsto m(z,x)>0 \tag{6.3}$$

are supposed to be Hoelder, and $W_1(t)$ and $W_2(t)$ are independent standard Wiener processes (WPs) in $\mathbb{R}$. We also specify that, for a diffusions with accessible boundaries we use the Neumann condition.

Accordingly, the generator $\mathbf{R}$ is a closed operator in $\mathrm{C_b}(\mathcal{Z}\times\mathbb{X})$; on C^2-functions $(z,x)\in\mathcal{Z}\times\mathbb{X}\mapsto\phi(z,x)$ its action is given by

$$\begin{aligned}\mathbf{R}\phi(z,x) &= a(x)\frac{\partial\phi}{\partial x}(z,x)+\frac{z^2}{2}\frac{\partial^2\phi}{\partial x^2}(z,x)\\ &+\frac{1}{m(z,x)}\left[c(z)\frac{\partial\phi}{\partial z}(z,x)+\frac{C(z)}{2}\frac{\partial^2\phi}{\partial z^2}(z,x)\right].\end{aligned}\tag{6.4}$$

More precisely, the domain $\mathrm{D}(\mathbf{R})$ is the closure of C^2-functions ϕ for which $\mathbf{R}\phi\in\mathrm{C_b}(\overline{\mathcal{Z}}\times\overline{\mathbb{X}})$, and the Neumann boundary conditions are fulfilled on the accessible parts of the boundary. Here $\overline{\mathcal{Z}}$ and $\overline{\mathbb{X}}$ stand for the closure of $\mathcal{Z}$ and $\mathbb{X}$ and $\mathrm{C_b}(\overline{\mathcal{Z}}\times\overline{\mathbb{X}})$ denotes the space of bounded continuous functions on $\overline{\mathcal{Z}}\times\overline{\mathbb{X}}$.

Viz., for $\mathbb{X}=\mathbb{R}$ and $\mathcal{Z}=\mathbb{R}_+=(0,\infty)$, assuming that 0 is an instant reflection point for $\widetilde{Z}(t)$, generator $\mathbf{R}$ introduced in (6.4) is defined on a domain $\mathrm{D}(\mathbf{R})$ which is the closure of $\mathrm{D}^0(\mathbf{R})$ where

$$\begin{aligned}\mathrm{D}^0(\mathbf{R})=\Big\{&\phi\in\mathrm{C}^2(\mathbb{R}_+\times\mathbb{R}):\ \mathbf{R}\phi\in\mathrm{C_b}(\overline{\mathbb{R}}_+\times\mathbb{R}),\\ &\mathbf{R}\phi(0+,x)=\lim_{z\to0+}\mathbf{R}\phi(z,x),\ \frac{\partial\phi(z,x)}{\partial z}\Big|_{z=0+}=0\ \ \forall\,x\in\mathbb{R}\Big\}.\end{aligned}\tag{6.5}$$

As was mentioned, in some examples the existence and uniqueness of a Feller diffusion $(Z(t),X(t))$ follows from general existence/uniqueness theorems; when such known results are not applicable, the notation $(Z(t),X(t))$ and the term a combined process have only an inspirational meaning, and the term IM is a euphemism for a solution to the WIE in Eqn (6.11).

To construct an IM $\boldsymbol{\kappa}$ for $(Z(t),X(t))$, we make two assumptions. First, we suppose that, $\forall$ given z, function $x\mapsto m(z,x)$ gives the density of an IM $\nu^{(z)}$ for a diffusion MP $\widetilde{X}^{(z)}(t)$ in $\mathbb{R}$ with some boundary conditions. The generator $\mathtt{L}^{(z)}$ of process $\widetilde{X}^{(z)}$ is a closed operator in $\mathrm{C_b}(\mathbb{X})$ whose action on C^2-functions $x\mapsto g(x)$ is given by

$$\mathtt{L}^{(z)}g(x)=a(x)g'(x)+\frac{z^2}{2}g''(x),\ \ x\in\mathbb{X}.\tag{6.6}$$

More precisely, the domain $\mathrm{D}(\mathtt{L}^{(z)})$ is the closure of the set of C^2-functions g for which $\mathtt{L}^{(z)}g\in\mathrm{C_b}(\overline{\mathbb{X}})$ and the derivative $g'(x)$ vanishes at the accessible points of the boundary $\partial\mathbb{X}$.

In other words, we assume that for a sufficient amount of functions g (forming a core for $\mathtt{L}^{(z)}$) we have that

$$\int_{\mathbb{D}}m(z,x)\,\mathtt{L}^{(z)}g(x)\mathrm{d}x=0,\ \ \text{implying}\ \ \mathtt{L}^{(z)^*}m(z,x)=0,\ \ z\in\mathcal{Z},x\in\mathbb{X}.\tag{6.7}$$

Here $\mathtt{L}^{(z)*}$ is the (properly defined) adjoint operator acting on Radon–Nikodym densities (relative to $\mathrm{d}x$). The reader familiar with the concepts of the scale and speed densities can think that

$$m(z,x) \propto \exp\left[\frac{2}{z^2}\int^x a(y)\mathrm{d}y\right].$$

Second, we assume that there is an SE diffusion MP $\widetilde{Z}(t)$ in interval $\mathcal{Z}$ which obeys the SDE

$$\mathrm{d}\widetilde{Z}(t) = c(\widetilde{Z}(t))\mathrm{d}t + [C(\widetilde{Z}(t))]^{1/2}\mathrm{d}\widetilde{W}_2(t), \quad \widetilde{Z}(0) = z_0, \tag{6.8}$$

and has an IM υ. (As above, the SDE (6.8) is subject to a modification at point 0 when $\mathcal{Z} = (0,\infty)$ and 0 is an accessible boundary.) Here $\widetilde{W}_2(t)$ is a standard Wiener process in $\mathbb{R}$, and $\mathrm{d}\upsilon(z)$ is assumed to be absolutely continuous wrt $\mathrm{d}z$: $\mathrm{d}\upsilon(z) = w(z)\mathrm{d}z$. The generator $\mathtt{A}$ of process $\widetilde{Z}(t)$ is a closed operator acting on C^2-functions $z \mapsto f(z)$ by

$$\mathtt{A}f(z) = c(z)f'(z) + \frac{C(z)}{2}f''(z), \quad z \in \mathcal{Z}; \tag{6.9}$$

its domain $\mathrm{D}(\mathtt{A})$ is the closure of the set of C^2-functions f with $\mathtt{A}f \in \mathrm{C_b}(\overline{\mathcal{Z}})$ such that $f' = 0$ at accessible endpoints of $\mathcal{Z}$. Thus, it is assumed that for sufficiently many functions f (forming a core for $\mathtt{A}$) we have that

$$\int_{\mathbb{B}} \mathtt{A}f(z)w(z)\mathrm{d}z = 0 \;\text{ implying }\; \mathtt{A}^*w(z) = 0, \quad z \in \mathbb{B}. \tag{6.10}$$

Referring, as before, to the scale and speed densities, one can assume that

$$w(z) \propto \exp\left[\int^z \frac{2c(y)}{C(y)}\mathrm{d}y\right].$$

We say that a measure $\boldsymbol{\eta}(\mathrm{d}z \times \mathrm{d}x)$ on $\mathcal{Z} \times \mathbb{X}$ satisfies the WIE with generator $\mathbf{R}$ if

$$\int_{\mathcal{Z}\times\mathbb{R}} \mathbf{R}\phi(z,x)\boldsymbol{\eta}(\mathrm{d}z \times \mathrm{d}x) = 0 \tag{6.11}$$

for any function $(z,x) \in \mathcal{Z} \times \mathbb{R} \mapsto \phi(z,x)$ satisfying conditions (i)–(ii) below. (i) $\forall\, z$, the section map $g_{\phi,z}:\; x \to \phi(z,x)$ lies in a core of operator $\mathtt{L}^{(z)}$. (ii) $\forall\, x$, the section map $f_{\phi,x}:\; z \to \phi(z,x)$ lies in a core of operator $\mathtt{A}$. (iii) The following integrals are finite:

$$\int_{\mathcal{Z}\times\mathbb{R}} \left| a(x)\frac{\partial\phi}{\partial x}(z,x) + \frac{z^2}{2}\frac{\partial^2\phi}{\partial x^2}(z,x)\right| \boldsymbol{\eta}(\mathrm{d}z \times \mathrm{d}x)$$

and

$$\int_{\mathcal{Z}\times\mathbb{R}} \frac{1}{m(z,x)}\left| c(z)\frac{\partial\phi}{\partial z}(z,x) + \frac{C(z)}{2}\frac{\partial^2\phi}{\partial z^2}(z,x)\right| \boldsymbol{\eta}(\mathrm{d}z \times \mathrm{d}x),$$

which allows us to use any order of integration in the summands emerging in the LHS of (6.11).

Theorem 6.1. *Assume* (6.7) *and* (6.10). *Then the following assertions hold true.* (I) *The measure* $\boldsymbol{\kappa}$ *with the Radon–Nikodym density*

$$\frac{\boldsymbol{\kappa}(\mathrm{d}z \times \mathrm{d}x)}{\upsilon(\mathrm{d}z) \times \mathrm{d}x} = m(z, x) \tag{6.12}$$

satisfies the WIE with generator $\mathbf{R}$ *from Eqn* (6.4). (II) *In case of an absolutely continuous measure* υ, *with* $\upsilon(\mathrm{d}z) = w(z)\mathrm{d}z$ *we obtain* $\dfrac{\boldsymbol{\kappa}(\mathrm{d}z \times \mathrm{d}x)}{\mathrm{d}z \times \mathrm{d}x} = m(z, x)w(z)$. (III) *Assume that* $\mathbb{X}$ *and* $\mathcal{Z}$ *are bounded intervals* $\mathbb{I} \subset \mathbb{R}$ *and* $\mathbb{J} \subset (\epsilon, \infty)$, *respectively, where* $\epsilon > 0$. *Suppose that coefficient functions* $a(x)$, $c(z)$, $C(z)$, $m(z, x)$ *are bounded and Hoelder, with* $\inf C(z) > 0$ *and* $\inf m(z, x) > 0$. *Then there is a unique Feller diffusuion process* $(Z(t), X(t))$ *with generator* $\mathbf{R}$ *in rectangle* $\mathbb{J} \times \mathbb{I}$, *and measure* $\boldsymbol{\kappa}$ *is an IM for* $(Z(t), X(t))$.

Proof. Let us start with (I). We have to check that $\int_{\mathcal{Z}\times\mathbb{R}} \big[\mathbf{R}\phi(z, x)\big]\boldsymbol{\kappa}(\mathrm{d}z \times \mathrm{d}x) = 0$, for each function $(z, x) \mapsto \phi(z, x)$ mentioned in the above definition of the WIE. We write that

$$\begin{aligned}\int_{\mathcal{Z}\times\mathbb{R}} \mathbf{R}\phi(z, x)\boldsymbol{\kappa}(\mathrm{d}z \times \mathrm{d}x) = \int_{\mathcal{Z}\times\mathbb{R}} \Bigg\{ &\left[a(x)\frac{\partial\phi}{\partial x}(z, x) + \frac{z^2}{2}\frac{\partial^2\phi}{\partial x^2}(z, x)\right] \\ &+\frac{1}{m(z, x)}\left[c(z)\frac{\partial\phi}{\partial z}(z, x) + \frac{C(z)}{2}\frac{\partial^2\phi}{\partial z^2}(z, x)\right]\Bigg\} m(z, x)\upsilon(\mathrm{d}z)\mathrm{d}x\end{aligned}$$

which is equal to a sum of integrals $I_1 + I_2$ where

$$\begin{aligned}I_1 &= \int \left[a(x)\frac{\partial\phi}{\partial x}(z, x) + \frac{z^2}{2}\frac{\partial^2\phi}{\partial x^2}(z, x)\right] m(z, x)\mathrm{d}\upsilon(z)\,\mathrm{d}x \\ &= \int_{\mathcal{Z}} \left\{\int_{\mathbb{D}} m(z, x)\left[\mathtt{L}^{(z)}\phi(z, \,\cdot\,)\right](x)\mathrm{d}x\right\} \mathrm{d}\upsilon(z)\end{aligned}$$

and

$$I_2 = \int \left[c(z)\frac{\partial\phi}{\partial z}(z, x) + \frac{C(z)}{2}\frac{\partial^2\phi}{\partial z^2}(z, x)\right] \upsilon(\mathrm{d}z)\mathrm{d}x = \int_{\mathbb{D}} \left[\int_{\mathcal{Z}} \left[\mathtt{A}\phi(\,\cdot\,, x)\right](z)\upsilon(\mathrm{d}z)\right] \mathrm{d}x.$$

In both expressions, the inner integrals vanish: in I_1 it occurs due to (6.7), and in I_2 by virtue of (6.10).

Assertion (II) is straightforward, whereas (III) follows from general results (see, e.g., [3], Section 8.1, Theorem 1.5 on P. 369).

We now pass to examples of interest.

6.B. We start with an example where $\mathcal{Z} = \mathbb{R}_+ = (0, \infty)$. Here, the environment MP $\widetilde{Z}(t)$ is represented by a BM in $\mathbb{R}_+$ with drift $b \in \mathbb{R}$ and reflection at 0. Recall: given a standard WP $W(t)$, there exists a solution to the Skorohod problem. That is, $\forall\, z_0 > 0$, $\exists$ a pair of processes $Z(t), L(t)$ such that (i) $Z(t) = z_0 + W(t) + L(t)$, $t \geq 0$, (ii) $Z(t) \geq 0$, $t \geq 0$, and (iii) process $L(t)$ has $L(0) = 0$ (starts

at 0), is continuous, monotone nondecreasing and obeys $\operatorname{supp}(\mathrm{d}L) \subset \{t \geq 0 : Z(t) = 0\}$ (that is, $L(t)$ increases only at times when $Z(t)$ is equal to zero). Moreover, (iv) $L(t) = [-z_0 - \inf_{s \leq t} W(s)]^+$. In addition, (v) the distributions of processes $Z(t)$ and $|W(t)|$ coincide. Formally, we have:

$$\begin{array}{ll} \mathtt{L}^{(z)}g(x) = -xg'(x) + \dfrac{z^2 g''(x)}{2}, \;\; m(z,x) = e^{-x^2/z^2}, & \\ \mathtt{A}f(z) = bf'(z) + \dfrac{f''(z)}{2}, \quad \mathrm{d}v(z) = e^{2bz}\mathrm{d}z, & \end{array} \quad x \in \mathbb{R}, z > 0. \tag{6.13}$$

The domain of operator $\mathtt{L}^{(z)}$ is the closure in $\mathrm{C_b}(\mathbb{R})$ of C^2-functions $x \in \mathbb{R} \mapsto g(x)$ such that $\mathtt{L}^{(z)}g \in \mathrm{C_b}(\mathbb{R})$. The domain of $\mathtt{A}$ is the closure in $\mathrm{C_b}(\mathbb{R}_+)$ of C^2-functions $z \in \mathbb{R}_+ \mapsto f(z)$ such that $f'(0+) = 0$ and $\mathtt{A}f \in \mathrm{C_b}(\mathbb{R}_+)$. Both $\mathtt{L}^{(z)}$ and $\mathtt{A}$ generate unique Feller semi-groups. For $\mathtt{L}^{(z)}$, the Feller semi-group is determined by the transition densities $p_t(x,x')$ from Eqn (6.2). For $\mathtt{A}$, the Feller semi-group is determined by the transition densities

$$\begin{aligned} p^{(t)}(z,z') \;&= \frac{2be^{2bz}}{e^{2bz}-1} + \frac{2}{\pi} e^{b(z'-z)-b^2t/2} \\ &\quad \times \int_0^\infty \frac{e^{-s^2t/2}}{s^2+b^2} \Big[s\cos(sz) + b\sin(sz) \Big] \Big[s\cos(sz') + b\sin(sz') \Big], \quad z, z' > 0. \end{aligned} \tag{6.14}$$

Cf. [15], Eqns (28)–(29) (more precisely, a displayed equation between (28) and (29).

In this example, for $x \in \mathbb{R}$ and $z > 0$ we have:

$$\begin{aligned} \mathbf{R}\phi(z,x) \;&= -x\frac{\partial}{\partial x}\phi(z,x) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}\phi(z,x) \\ &\quad + e^{x^2/z^2}\left[b\frac{\partial}{\partial z}\phi(z,x) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\phi(z,x) \right], \end{aligned} \tag{6.15}$$

and

$$\mathrm{d}\boldsymbol{\kappa}(z,x) = e^{2bz - x^2/z^2}\mathrm{d}x\,\mathrm{d}z. \tag{6.16}$$

The domain $\mathrm{D}(\mathbf{R})$ of operator $\mathbf{R}$ in (6.15) is the closure of the set $\mathrm{D}^0(\mathbf{R})$ from (6.5). Due to presence of function $e^{x^2/z^2} = [m(z,x)]^{-1}$, the existing general results do not guarantee that operator $\mathbf{R}$ generates a unique Feller semi-group in $\mathrm{C_b}(\mathbb{R}_+ \times \mathbb{R})$. Nevertheless, Theorem 8 (I) holds, and $\boldsymbol{\kappa}$ satisfies the WIE with $\mathbf{R}$.

However, if we take $\mathbb{X}$ to be an interval $\mathbb{I} = (x^1, x^2)$ and $\mathcal{Z} = (z^1, z^2)$, with $-\infty < x^1 < x^2 < \infty$ and $0 < z^1 < z^2 < \infty$, and assume that both $\widetilde{X}^{(z)}$ and $\widetilde{Z}(t)$ are reflected at the endpoints x^i and z^i then the situation changes. We still define $\mathbf{R}$ and $\boldsymbol{\kappa}$ via Eqns (6.15) and (6.16), for $x \in \mathbb{I}$, $z \in \mathbb{J}$, but the domain $\mathrm{D}(\mathbf{R})$ involves the Neumann boundary conditions $\dfrac{\partial}{\partial x}\phi(z,x^i) = 0$ and $\dfrac{\partial}{\partial z}\phi(z^i,x) = 0$. Under these modifications, operator $\mathbf{R}$ generates a unique Feller semi-group in $\mathrm{C_b}(\mathbb{J} \times \mathbb{I})$. Furthermore, by Proposition 9.2 in [3], P. 239, $\boldsymbol{\kappa}$ is a genuine IM for this sem-group.

The corresponding MP is a pair $(X(t), Z(t))$ plus an auxiliary quadruple $(L_X(t), U_X(t), L_Z(t), U_Z(t))$ solving a system of SIDEs

$$\begin{gathered}
\mathrm{d}X(t) = -X(t)\mathrm{d}t + Z(t)\mathrm{d}W_1(t) + \mathrm{d}L_X(t) - \mathrm{d}U_X(t),\ X(0) = x^0 \in (x_1, x_2) \\
\mathrm{d}Z(t) = -be^{\frac{X^2(t)}{Z^2(t)}}\mathrm{d}t + e^{\frac{X^2(t)}{2Z^2(t)}}\mathrm{d}W_2(t) + \mathrm{d}L_Z(t) - \mathrm{d}U_Z(t),\quad Z(0) = z^0 \in (z_1, z_2), \\
\int_0^t \mathbf{1}(X(s) = x_1)\mathrm{d}L_X(s) = L_X(t),\quad \int_0^t \mathbf{1}(X(s) = x_2)\mathrm{d}U_X(s) = U_X(t), \\
\int_0^t \mathbf{1}(Z(s) = z_1)\mathrm{d}L_Z(s) = L_Z(t),\quad \int_0^t \mathbf{1}(Z(s) = z_2)\mathrm{d}U_Z(s) = U_Z(t),
\end{gathered}$$

where $L_X(t), U_X(t)$ are the local times spend by the process $X(t)$ at points x_1 and x_2 and $L_Z(t), U_Z(t)$ are local times spend by the process $Z(t)$ at points z_1 and z_2.

Here and below, x_0, z_0 are initial values and $W_1(t)$, $W_2(t)$ are independent WPs in $\mathbb{R}$.

6.C. In this example we take $\mathcal{Z} = \mathbb{R}$ and deal with an OUP $\widetilde{X}^{(z)}(t)$ where the diffusion coefficient z follows its own OUP. Formally speaking, we set

$$\begin{array}{l}
\mathtt{L}^{(z)}g(x) = -xg'(x) + \dfrac{z^2}{2}g''(x),\ \ m(z,x) = e^{-x^2/z^2}, \\
\mathtt{A}f(z) = -zf'(z) + \dfrac{1}{2}f''(z),\ \ \mathrm{d}v(z) = e^{-z^2}\mathrm{d}z,
\end{array} \qquad z, x \in \mathbb{R}. \tag{6.17}$$

Here, the Feller semi-group generated by $\mathtt{A}$ is determined by the transition densities similar to (6.2):

$$p^{(t)}(z, z') = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{(z' - ze^{-t})^2}{(1 - e^{-2t})}\right],\ \ z, z' \in \mathbb{R}. \tag{6.18}$$

Then, for $z, x \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{R}\phi(z,x) \ &= -x\frac{\partial}{\partial x}\phi(z,x) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}\phi(z,x) \\
&\qquad + e^{x^2/z^2}\left[-z\frac{\partial}{\partial z}\phi(z,x) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\phi(z,x)\right],
\end{aligned} \tag{6.19}$$

and

$$\mathrm{d}\boldsymbol{\kappa}(z,x) = e^{-z^2 - x^2/z^2}\mathrm{d}x\,\mathrm{d}z. \tag{6.20}$$

The domain $\mathrm{D}(\mathbf{R})$ is the closure, in $\mathrm{C_b}(\mathbb{R}\times\mathbb{R})$, of C^2-functions $(z,x) \in \mathbb{R}\times\mathbb{R} \mapsto \phi(z,x)$ for which $\mathbf{R}\phi \in \mathrm{C_b}(\mathbb{R}\times\mathbb{R})$. (The boundary condition in (6.19) is omitted.) Again, the available general results do not allow us to conclude that there exists a unique Feller semi-group generated by $\mathbf{R}$, but Theorem 8 (I) holds true.

However, as before, we can take $\mathbb{X} = (x^1, x^2) := \mathbb{I}$ and $\mathcal{Z} = (z^1, z^2) := \mathbb{J}$ where $-\infty < x^1 < x^2 < \infty$, $-\infty < z^1 < z^2 < \infty$ and $0 \notin \mathbb{J}$, and use Eqns (6.19) and (6.20) for $z \in \mathbb{J}$, $x \in \mathbb{I}$, adding the Neumann boundary conditions $\dfrac{\partial}{\partial x}\phi(z.x^i) = 0$ and $\dfrac{\partial}{\partial z}\phi(z^i.x) = 0$ in the definition of $\mathrm{D}(\mathbf{R})$. Then $\mathbf{R}$ generates a unique Feller semi-group in $\mathrm{C}(\mathbb{J}\times\mathbb{I})$, and $\boldsymbol{\kappa}$ is a genuine IM.

The corresponding diffusion MP is a pair $(X(t), Z(t))$ from the solution to SIDEs

$$
\begin{array}{c}
\mathrm{d}X(t) = -X(t)\mathrm{d}t + Z(t)\mathrm{d}W_1(t) + \mathrm{d}L_X(t) - \mathrm{d}U_X(t),\ X(0) = x^0 \in (x_1, x_2),\\
\mathrm{d}Z(t) = -Z(t)e^{\frac{X^2(t)}{Z^2(t)}}\mathrm{d}t + e^{\frac{X^2(t)}{2Z^2(t)}}\mathrm{d}W_2(t) + \mathrm{d}L_Z(t) - \mathrm{d}U_Z(t),\ Z(0) = z^0 \in (z_1, z_2),\\
\int_0^t \mathbf{1}(X(s) = x_1)\mathrm{d}L_X(s) = L_X(t), \quad \int_0^t \mathbf{1}(X(s) = x_2)\mathrm{d}U_X(s) = U_X(t),\\
\int_0^t \mathbf{1}(Z(s) = z_1)\mathrm{d}L_Z(s) = L_Z(t), \quad \int_0^t \mathbf{1}(Z(s) = z_2)\mathrm{d}U_Z(s) = U_Z(t).
\end{array}
$$

6.D. Finally, consider the case where the diffusion coefficient z follows a Cox–Ingersoll–Ross process on $\mathbb{R}_+$: see [2]. This example can be considered as a modification of the Heston model of a stochastic volatility; cf. [7]. Here we start with $\mathcal{Z} = \mathbb{R}_+$ and generator $\mathtt{A}$ of the SE diffusion $\widetilde{Z}(t)$ of the form

$$
\mathtt{A}f(z) = a(b-z)f'(z) + \frac{z}{2}f''(z),\ \ \mathrm{d}\upsilon(z) = z^{2ab-1}e^{-2az}\mathrm{d}z,\ z > 0, \tag{6.21}
$$

where $a \geq 0, b > 0$ are given parameters. The domain $\mathrm{D}(\mathtt{A})$ consists of C^2-functions $z \in \mathbb{R}_+ \mapsto f(z)$ such that $\mathtt{A}f \in \mathrm{C_b}(\mathbb{R}_+)$ and – when $a > 1/2$ – the right derivative $f'(0+) = 0$. The Feller semigroup generated by $\mathtt{A}$ has the transition density

$$
p^{(t)}(z, z') = c\Big[\exp\big(-cze^{-at} - cz'\big)\Big]\left(\frac{z'e^{at}}{z}\right)^{q/2} I_q\left(2c\sqrt{zz'e^{-at}}\right),\ \ z, z' > 0, \tag{6.22}
$$

where $c = \dfrac{2a}{1-e^{-at}}$, $q = 2ab - 1$ and I_q is the Bessel function of order q. (For $a \leq 1/2$, process $\widetilde{Z}(t)$ does not hit 0, whereas for $a > 1/2$ it hits 0 at infinitely many times that are indefinitely large.)

The basic process is, as before, an OUP, with generator $\mathtt{L}^{(z)}$, as in (6.1), (6.13) and (6.17). For $x \in \mathbb{R}$, $z > 0$, the combined generator is

$$
\begin{aligned}
\mathbf{R}\phi(z, x) \ &= -x\frac{\partial}{\partial x}\phi(z, x) + \frac{z^2}{2}\frac{\partial^2}{\partial x^2}\phi(z, x)\\
&\quad + e^{x^2/z^2}\left[a(b-z)\frac{\partial}{\partial z}\phi(z, x) + \frac{z}{2}\frac{\partial^2}{\partial z^2}\phi(z, x)\right],
\end{aligned} \tag{6.23}
$$

and we set

$$
\mathrm{d}\boldsymbol{\kappa}(z, x) = z^{2ab-1}e^{-2az - x^2/z^2}\mathrm{d}x\,\mathrm{d}z. \tag{6.24}
$$

The domain $\mathrm{D}(\mathbf{R})$ is the closure of $\mathrm{D}^0(\mathbf{R})$ where

$$
\begin{aligned}
\mathrm{D}^0(\mathbf{R}) = \Big\{\phi \in \mathrm{C}^2(\mathbb{R}_+ \times \mathbb{R}) :\ \ &\mathbf{R}\phi \in \mathrm{C_b}(\overline{\mathbb{R}}_+ \times \mathbb{R}) \text{ and – for } a > 1/2 \text{ – also}\\
&\frac{\partial}{\partial z}\phi(z, x)\Big|_{z=0+} = 0 \ \text{ and } \ \lim_{z \to 0+} \mathbf{R}\phi(z, x) = \mathbf{R}\phi(0+, x)\Big\};
\end{aligned} \tag{6.25}
$$

cf. (6.5)). Again we can't derive that there is a unique Feller semi-group generated by $\mathbf{R}$ but Theorem 8 (I) holds true.

Taking $\mathbb{X} = (x^1, x^2)$ and $\mathcal{Z} = (z^1, z^2)$ with $-\infty < x^1 < x^2 < \infty$ and $0 < z^1 < z^2 < \infty$ and introducing Neumann boundary conditions leads, as above, to a unique Feller semi-group for which $\boldsymbol{\kappa}$ is a genuine IM. The coresponding diffusion MP is a pair $(X(t), Z(t))$ solving, together with $(L_X(t), U_X(t), L_Z(t), U_Z(t))$, the system

$$\begin{gathered}
\mathrm{d}X(t) = X(t)\mathrm{d}t + Z(t)\mathrm{d}W_1(t) + \mathrm{d}L_X(t) - \mathrm{d}U_X(t),\ X(0) = x^0 \in (x_1, x_2) \\
\mathrm{d}Z(t) = a(b - Z(t))e^{\frac{X^2(t)}{Z^2(t)}}\mathrm{d}t + \sqrt{Z(t)}e^{\frac{X^2(t)}{2Z^2(t)}}\mathrm{d}W_2(t) + \mathrm{d}L_Z(t) - \mathrm{d}U_Z(t),\ Z(0) = z^0 \in (z_1, z_2), \\
\int_0^t \mathbf{1}(X(s) = x_1)\mathrm{d}L_X(s) = L_X(t), \quad \int_0^t \mathbf{1}(X(s) = x_2)\mathrm{d}U_X(s) = U_X(t), \\
\int_0^t \mathbf{1}(Z(s) = z_1)\mathrm{d}L_Z(s) = L_Z(t), \quad \int_0^t \mathbf{1}(Z(s) = z_2)\mathrm{d}U_Z(s) = U_Z(t).
\end{gathered}$$

A similar construction can be done if we consider the SE process $\widetilde{Z}(t)$ on an interval $(0, r)$ instead of $\mathbb{R}_+$; cf. Section 6.1 in Ref. [15].

Remarks. 6.1. One of intriguing problems emerging from the above constuction (in its general form as well as in specific examples) is to solve (and in fact, to pose in a correct manner) an inverse problem. (We can speak of a hidden Markov model.) Informally, we ask: is it possible to represent a given a random process $X(t)$ as a projection $(Z(t), X(t)) \mapsto X(t)$ of a combined MP $(Z(t), X(t))$ obtained by means of the construction, where component $Z(t)$ describes a dynamics of the SE? E.g., in the context of sub-section 4.A: can one represent a birth-death process on $\mathbb{Z}_+$ as a result of a projection $(z, n) \in \mathcal{Z} \times \mathbb{Z}_+ \mapsto n$, from an MP on $\mathcal{Z} \times \mathbb{Z}_+$ constructed in the above fashion? A similar question in the context of sub-sections 4.B is: given a process with continuous paths on (a domain $\mathbb{D}$ in) $\mathbb{R}^d$, is it possible to represent it as a projection of an MP in a higher dimension, obtained by means of the above construction? In terms of Section 5, when it is possible to represent a given process with continuous paths on (a domain $\mathbb{D}$ in) $\mathbb{R}^d$ as a projection of a diffusion in higher dimension?

6.2. A feature of he construction presented in the paper is that the factor $m(z, x)$ (and also $\sigma(z)$ in previous parts of the paper) enter both the generator $\mathbf{R}$ and IM $\boldsymbol{\kappa}$ in a mutually inverse fashion. It creates an interesting pattern: if $m(z, x)$ is small then we have growing coefficients in $\mathbf{R}$ causing difficulties with proving the existence of the combined MP $(Z(t), X(t))$. But then we get an IM $\boldsymbol{\kappa}$ which may be made an EPD (at least, at a formal level of the corresponding WIE). This suggests that for the models under consideration some new kind of existence theorems are possible, under weaker assumptions than in (5.11). This may present an interesting direction for future development.

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